

Zero sets of eigenspinors for generic metrics <sup>\*</sup>

Andreas Hermann

**Abstract**

Let  $M$  be a closed connected spin manifold of dimension 2 or 3 with a fixed orientation and a fixed spin structure. We prove that for a generic Riemannian metric on  $M$  the non-harmonic eigenspinors of the Dirac operator are nowhere zero. The proof is based on a transversality theorem and the unique continuation property of the Dirac operator.

**Contents**

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Preliminaries</b>	<b>4</b>
2.1	Spinors for different metrics . . . . .	4
2.2	Further preliminaries . . . . .	5
<b>3</b>	<b>Green's function for the Dirac operator</b>	<b>7</b>
3.1	The Bourguignon-Gauduchon trivialization . . . . .	7
3.2	The Euclidean Dirac operator . . . . .	9
3.3	Expansion of Green's function . . . . .	11
<b>4</b>	<b>Zero sets of eigenspinors</b>	<b>18</b>
4.1	Eigenspinors in dimensions 2 and 3 . . . . .	18
4.2	Examples on closed surfaces . . . . .	31

**1 Introduction**

Let  $(M, g)$  be an oriented Riemannian spin manifold of dimension  $n$ . We denote by  $P_{\text{SO}}(M, g)$  the principal  $\text{SO}(n)$ -bundle of positively oriented  $g$ -orthonormal frames. A spin structure on  $(M, g)$  consists of a principal

---

<sup>\*</sup>AMS Subject classification numbers: 53C27 (primary), 58J05, 35J08 (secondary)

$\text{Spin}(n)$ -bundle  $P_{\text{Spin}}(M, g)$  and a two-fold covering

$$\Theta : P_{\text{Spin}}(M, g) \rightarrow P_{\text{SO}}(M, g),$$

which is compatible with the group actions of  $\text{SO}(n)$  on  $P_{\text{SO}}(M, g)$  and of  $\text{Spin}(n)$  on  $P_{\text{Spin}}(M, g)$ . We will always assume that a spin manifold has a fixed orientation and a fixed spin structure. If in addition a Riemannian metric  $g$  on  $M$  is chosen, we denote the Riemannian spin manifold by  $(M, g, \Theta)$ . The spinor bundle is defined as the associated vector bundle  $\Sigma^g M := P_{\text{Spin}}(M, g) \times_{\rho} \Sigma_n$ , where  $\rho$  is the spinor representation on the complex vector space  $\Sigma_n$  of dimension  $2^{[n/2]}$ . The Dirac operator  $D^g$  is a linear elliptic differential operator of first order acting on smooth sections of  $\Sigma^g M$ :

$$D^g : C^\infty(\Sigma^g M) \rightarrow C^\infty(\Sigma^g M).$$

A detailed introduction to the concepts of spin geometry, which we use here, is given in [LM].

In some questions arising in conformal spin geometry and in mathematical physics the zero sets of solutions to generalized Dirac equations play an important role. Given a compact Riemannian spin manifold  $(M, g, \Theta)$  one would like to obtain bounds on the eigenvalues of  $D^g$  which are uniform in the conformal class  $[g]$  of  $g$ . For every metric  $h \in [g]$  let  $\lambda_1^+(h)$  be the smallest positive eigenvalue of  $D^h$  and let  $\lambda_1^-(h)$  be its largest negative eigenvalue. The two conformal invariants

$$\lambda_{\min}^\pm := \inf_{h \in [g]} |\lambda_1^\pm(h)| \text{vol}(M, h)^{1/n}$$

have been studied by many authors. A non-exhaustive list is [Hij1], [Hij2], [Lo], [Bä1], [Am2]. A natural question is whether the infimum is attained at a Riemannian metric. By a result of B. Ammann this is the case, if the nonlinear partial differential equation

$$D^g \psi = \lambda_{\min}^+ |\psi|_g^{2/(n-1)} \psi, \quad \|\psi\|_{2n/(n-1)} = 1 \quad (1)$$

has a solution  $\psi$ , which is nowhere zero on  $M$  (see [Am4]). It is not obvious that a solution without zeros exists.

Another example involves the zero sets of Witten spinors. Originally these spinors were used by E. Witten to prove the positive energy theorem for spin manifolds (see [Wi]). But it has also been suggested to use them in order to construct special orthonormal frames of the tangent bundle of an asymptotically flat manifold of dimension 3 (see [N], [DM], [FNS]). It turns

out that this is possible, if one can find a Witten spinor which is nowhere zero. However it is not clear that such a spinor exists.

In this article we consider the dependence of the zero sets of eigenspinors of the Dirac operator on the metric. We restrict to the case of closed Riemannian spin manifolds but we hope that some of the techniques developed here may be carried over to the questions mentioned above. The spectrum of the Dirac operator has been computed explicitly for some Riemannian spin manifolds. The result on the round sphere (see e.g. [Bä3]) shows that the multiplicity of an eigenvalue can be greater than the rank of the spinor bundle, which implies that there exist eigenspinors with non-empty zero set for special choices of a Riemannian metric. However it is also known ([Da]) that for a generic Riemannian metric on a closed spin manifold of dimension 2 or 3 the non-zero eigenvalues are simple (see Section 4.1 for a precise definition of the term “generic”). Thus one might conjecture that in the generic case the eigenvectors corresponding to non-zero eigenvalues should also be nowhere zero. We prove that this is indeed true. More precisely let  $M$  be a closed spin manifold and denote by  $R(M)$  the set of all smooth Riemannian metrics on  $M$ . For every  $g \in R(M)$  denote by  $[g] \subset R(M)$  the conformal class of  $g$ . Furthermore let  $N(M)$  be the set of all  $g \in R(M)$  such that all the non-harmonic eigenspinors of  $D^g$  are nowhere zero on  $M$ . Then we prove the following.

**Theorem 1.1.** *Let  $M$  be a closed connected spin manifold of dimension 2 or 3 with a fixed orientation and a fixed spin structure. Then for every  $g \in R(M)$  the set  $N(M) \cap [g]$  is residual in  $[g]$  with respect to every  $C^k$ -topology,  $k \geq 1$ .*

Recall that a subset is residual, if it contains a countable intersection of open and dense sets. In Section 4.2 we will give an example showing that in dimension 2 an analogue of this theorem for harmonic spinors does not hold. Furthermore it is known that for  $n = 3$  the set of all Riemannian metrics  $g$  satisfying  $\ker D^g = 0$  is generic (see [Ma]). Thus for  $n = 3$  the assertion of our theorem holds for all eigenspinors, not only the non-harmonic ones.

The main idea of the proof of Theorem 1.1 is as follows. If  $g, h$  are two Riemannian metrics on the spin manifold  $M$ , then a natural isomorphism between the two vector bundles  $\Sigma^g M$  and  $\Sigma^h M$  is well known. We construct a continuous map  $F$  defined on a suitable space of Riemannian metrics, which associates to every metric  $h$  an eigenspinor of the corresponding Dirac operator  $D^h$  viewed as a section of  $\Sigma^g M$ . Theorem 1.1 then follows from a transversality theorem. In order to apply this theorem we have to make sure that the evaluation map corresponding to  $F$  is transverse to the zero section

of  $\Sigma^g M$ . Our assumption that this is not the case leads to an equation involving Green's function for the operator  $D^g - \lambda$  with  $\lambda \in \mathbb{R}$ . From the expansion of this Green's function we obtain a contradiction using the unique continuation property of the Dirac operator.

This article is organized as follows. In Section 2.1 we recall a method of identifying spinors for different metrics, while in Section 2.2 we state a transversality theorem and a unique continuation theorem, which are both used in the proof. Section 3 contains the derivation of the expansion of Green's function for the operator  $D^g - \lambda$  around the singularity with  $\lambda \in \mathbb{R}$ . The proof of Theorem 1.1 is then carried out in Section 4.1, while in Section 4.2 we give an example showing that the theorem does not hold for harmonic spinors on closed surfaces.

**Acknowledgements** The author wishes to thank Bernd Ammann (Regensburg) and Mattias Dahl (Stockholm) for many interesting discussions.

## 2 Preliminaries

### 2.1 Spinors for different metrics

Let  $(M, g, \Theta)$  be a closed Riemannian spin manifold. Since the spinor bundles  $\Sigma^g M$  and  $\Sigma^h M$  are two different vector bundles, the question arises how one can identify spinors on  $(M, g)$  with spinors on  $(M, h)$  in a natural way. The case of conformally related metrics  $g$  and  $h$  has been treated in [Hit], [Hij1]. For general Riemannian metrics  $g$  and  $h$  Bourguignon and Gauduchon [BG] have solved this problem. In this article we only need the case of conformally related metrics.

Let  $h = e^{2u}g$  with  $u \in C^\infty(M, \mathbb{R})$ . Then by [Hit], [Hij1] there is an isomorphism

$$\beta_{g,h} : \Sigma^g M \rightarrow \Sigma^h M$$

of vector bundles, which is a fiberwise isometry. Then for all  $\psi \in C^\infty(\Sigma^g M)$  we have

$$D^h(e^{-(n-1)u/2}\beta_{g,h}\psi) = e^{-(n+1)u/2}\beta_{g,h}D^g\psi. \quad (2)$$

Furthermore, if  $h, k \in [g]$ , then the construction in [Hij1] shows that we have

$$\beta_{g,k} = \beta_{h,k} \circ \beta_{g,h}. \quad (3)$$

This formula is also derived in [BG] together with the fact that an analogue for general Riemannian metrics  $g, h, k$  does not hold. Our aim is to compare the Dirac operator  $D^h$  to  $D^g$  using a differential operator

$$D^{g,h} : C^\infty(\Sigma^g M) \rightarrow C^\infty(\Sigma^g M).$$

In order to obtain an operator with a self-adjoint closure in  $L^2(\Sigma^g M)$  one also has to take into account the different volume forms  $\mathrm{d}v^g$  and  $\mathrm{d}v^h$  induced by the metrics  $g$  and  $h$ , as was remarked in S. Maier's article [Ma]. Namely using that  $\mathrm{d}v^h = e^{nu} \mathrm{d}v^g$  and defining

$$\bar{\beta}_{g,h} := e^{-nu/2} \beta_{g,h} : \quad \Sigma^g M \rightarrow \Sigma^h M.$$

we see that the operator

$$D^{g,h} := \bar{\beta}_{h,g} D^h \bar{\beta}_{g,h}$$

has a self-adjoint closure in  $L^2(\Sigma^g M)$ . Furthermore for all  $\psi \in C^\infty(\Sigma^g M)$  we have

$$D^{g,h} \psi = e^{-u/2} D^g (e^{-u/2} \psi)$$

by the equation (2).

Let  $f \in C^\infty(M, \mathbb{R})$  and let  $I \subset \mathbb{R}$  be an open interval containing 0 such that for all  $t \in I$  the tensor field  $g_t := g + tfg$  is a Riemannian metric on  $M$ . Then for all  $\psi \in C^\infty(\Sigma^g M)$  we have

$$D^{g,g_t} \psi = (1 + tf)^{-1/4} D^g ((1 + tf)^{-1/4} \psi)$$

and therefore

$$\frac{d}{dt} D^{g,g_t} \Big|_{t=0} \psi = -\frac{1}{2} f D^g \psi - \frac{1}{4} \operatorname{grad}^g(f) \cdot \psi. \quad (4)$$

Furthermore the family  $(g_t)_{t \in I}$  of Riemannian metrics on  $M$  induces real analytic families of eigenvalues and eigenspinors of the family of Dirac operators  $(D^{g,g_t})_{t \in I}$ . Namely by a theorem of Rellich (see Thm. VII.3.9 in [K], [BG]) we have the following lemma.

**Lemma 2.1.** *Let  $\lambda$  be an eigenvalue of  $D^g$  with  $d := \dim_{\mathbb{C}} \ker(D^g - \lambda)$ . Then there exist real analytic functions  $\lambda_1, \dots, \lambda_d$  on  $I$  such that  $\lambda_j(t)$  is an eigenvalue of  $D^{g,g_t}$  for all  $j$  and all  $t$  and  $\lambda_j(0) = \lambda$  for all  $j$ . Furthermore there exist spinors  $\psi_{j,t}$ ,  $1 \leq j \leq d$ ,  $t \in I$ , which are real analytic in  $t$ , such that for every  $t \in I$  the spinors  $\psi_{j,t}$ ,  $1 \leq j \leq d$ , form an  $L^2$ -orthonormal system and for every  $t \in I$  and for every  $j$  the spinor  $\psi_{j,t}$  is an eigenspinor of  $D^{g,g_t}$  corresponding to  $\lambda_j(t)$ .*

## 2.2 Further preliminaries

In this section we briefly recall a transversality theorem from differential topology and a unique continuation theorem for Laplace type operators acting on sections of a vector bundle. Both theorems will be crucial for the proof of our main theorem.

**Definition 2.2.** Let  $f: Q \rightarrow N$  be a  $C^1$  map between two manifolds. Let  $A \subset N$  be a submanifold.  $f$  is called *transverse to  $A$* , if for all  $x \in Q$  with  $f(x) \in A$  we have

$$T_{f(x)}A + \text{im}(df|_x) = T_{f(x)}N.$$

We quote the following transversality theorem from [Hir], [U].

**Theorem 2.3.** Let  $V, M, N$  be smooth manifolds and let  $A \subset N$  be a smooth submanifold. Let  $F: V \rightarrow C^r(M, N)$  be a map, such that the evaluation map  $F^{ev}: V \times M \rightarrow N, (v, m) \mapsto F(v)(m)$  is  $C^r$  and transverse to  $A$ . If

$$r > \max\{0, \dim M + \dim A - \dim N\},$$

then the set of all  $v \in V$ , such that the map  $F(v)$  is transverse to  $A$ , is residual and therefore dense in  $V$ .

Let  $(M, g)$  be a Riemannian manifold and let  $\Sigma$  be a vector bundle over  $M$  with a connection  $\nabla$ . Then the connection Laplacian

$$\nabla^*\nabla: C^\infty(\Sigma) \rightarrow C^\infty(\Sigma)$$

is a linear elliptic differential operator of second order. In terms of a local  $g$ -orthonormal frame  $(e_i)_{i=1}^n$  of  $TM$  it is given by

$$\nabla^*\nabla\psi = -\sum_{i=1}^n \nabla_{e_i} \nabla_{e_i} \psi + \sum_{i=1}^n \nabla_{\nabla_{e_i} e_i} \psi. \quad (5)$$

We will use the following unique continuation theorem due to Aronszajn ([Ar], quoted from [Bä3]).

**Theorem 2.4.** Let  $(M, g)$  be a connected Riemannian manifold and let  $\Sigma$  be a vector bundle over  $M$  with a connection  $\nabla$ . Let  $P$  be an operator of the form  $P = \nabla^*\nabla + P_1 + P_0$  acting on sections of  $\Sigma$ , where  $P_1, P_0$  are differential operators of order 1 and 0 respectively. Let  $\psi$  be a solution to  $P\psi = 0$ . If there exists a point, at which  $\psi$  and all derivatives of  $\psi$  of any order vanish, then  $\psi$  vanishes identically.

If  $(M, g, \Theta)$  is a closed Riemannian spin manifold, then from the connection  $\nabla^g$  on the spinor bundle one obtains a connection Laplacian  $\nabla^*\nabla$ . For all  $\psi \in C^\infty(\Sigma^g M)$  we have by the Schrödinger-Lichnerowicz formula

$$(D^g)^2\psi = \nabla^*\nabla\psi + \frac{\text{scal}^g}{4}\psi \quad (6)$$

(see [LM] p. 160), where  $\text{scal}^g$  is the scalar curvature of  $(M, g)$ . Thus the above theorem applies to the operator  $(D^g)^2$ .

### 3 Green's function for the Dirac operator

#### 3.1 The Bourguignon-Gauduchon trivialization

Let  $(M, g, \Theta)$  be a Riemannian spin manifold of dimension  $n$ . In this section we explain a certain local trivialization of the spinor bundle  $\Sigma^g M$ , which we will use later. In the literature (e.g. [AGHM], [AH]) it is known as the Bourguignon-Gauduchon trivialization.

Let  $p \in M$ , let  $U$  be an open neighborhood of  $p$  in  $M$  and let  $V$  be an open neighborhood of 0 in  $\mathbb{R}^n$ , such that there exists a local parametrization  $\rho: V \rightarrow U$  of  $M$  by Riemannian normal coordinates with  $\rho(0) = p$ . The spinor bundle over the Euclidean space  $(\mathbb{R}^n, g_{\text{eucl}})$  with the unique spin structure will be denoted by  $\Sigma \mathbb{R}^n$ .

Let  $x \in V$ . Then there exists  $G_x \in \text{End}(T_x V)$ , such that for all vectors  $v, w \in T_x V$  we have

$$(\rho^* g)(v, w) = g_{\text{eucl}}(G_x v, w)$$

and  $G_x$  is  $g_{\text{eucl}}$ -self-adjoint and positive definite. There is a unique positive definite endomorphism  $B_x \in \text{End}(T_x V)$  such that we have  $B_x^2 = G_x^{-1}$ . If  $(E_i)_{i=1}^n$  is any  $g_{\text{eucl}}$ -orthonormal basis of  $T_x V$ , then  $(B_x E_i)_{i=1}^n$  is a  $\rho^* g$ -orthonormal basis of  $T_x V$ . Therefore the vectors  $d\rho|_x B_x E_i$ ,  $1 \leq i \leq n$ , form a  $g$ -orthonormal basis of  $T_{\rho(x)} M$ . We assemble the maps  $B_x$  to obtain a vector bundle endomorphism  $B$  of  $TV$  and we define

$$b: TV \rightarrow TM|_U, \quad b = d\rho \circ B.$$

From this we obtain an isomorphism of principal  $\text{SO}(n)$ -bundles

$$P_{\text{SO}}(V, g_{\text{eucl}}) \rightarrow P_{\text{SO}}(U, g), \quad (E_i)_{i=1}^n \mapsto (b(E_i))_{i=1}^n,$$

which lifts to an isomorphism of principal  $\text{Spin}(n)$ -bundles

$$c: P_{\text{Spin}}(V, g_{\text{eucl}}) \rightarrow P_{\text{Spin}}(U, g).$$

We define

$$\beta: \Sigma \mathbb{R}^n|_V \rightarrow \Sigma^g M|_U, \quad [s, \sigma] \mapsto [c(s), \sigma].$$

This gives an identification of the spinor bundles, which is a fibrewise isometry with respect to the bundle metrics on  $\Sigma \mathbb{R}^n|_V$  and on  $\Sigma^g M|_U$ . Furthermore for all  $X \in TV$  and for all  $\varphi \in \Sigma \mathbb{R}^n|_V$  we have  $\beta(X \cdot \varphi) = b(X) \cdot \beta(\varphi)$ . We obtain an isomorphism

$$A: C^\infty(\Sigma^g M|_U) \rightarrow C^\infty(\Sigma \mathbb{R}^n|_V), \quad \psi \mapsto \beta^{-1} \circ \psi \circ \rho,$$

which sends a spinor on  $U$  to the corresponding spinor in the trivialization. Let  $\nabla^g$  and  $\nabla$  denote the Levi Civita connections on  $(U, g)$  resp. on  $(V, g_{\text{eucl}})$  as well as its lifts to  $\Sigma^g M|_U$  and  $\Sigma \mathbb{R}^n|_V$ . Let  $(e_i)_{i=1}^n$  be a positively oriented orthonormal frame of  $TM|_U$ . Then for all  $\psi \in C^\infty(\Sigma^g M|_U)$  we have

$$\nabla_{e_i}^g \psi = \partial_{e_i} \psi + \frac{1}{4} \sum_{j,k=1}^n \tilde{\Gamma}_{ij}^k e_j \cdot e_k \cdot \psi,$$

where

$$\tilde{\Gamma}_{ij}^k := g(\nabla_{e_i}^g e_j, e_k)$$

(see [LM], p. 103, 110). In particular we can take the standard basis  $(E_i)_{i=1}^n$  of  $\mathbb{R}^n$  and put  $e_i := b(E_i)$ ,  $1 \leq i \leq n$ . We define the matrix coefficients  $B_i^j$  by  $B(E_i) = \sum_{j=1}^n B_i^j E_j$ . It follows that

$$\begin{aligned} A \nabla_{e_i}^g \psi &= \nabla_{d\rho^{-1}(e_i)} A\psi + \frac{1}{4} \sum_{j,k=1}^n \tilde{\Gamma}_{ij}^k E_j \cdot E_k \cdot A\psi \\ &= \nabla_{E_i} A\psi + \sum_{j=1}^n (B_i^j - \delta_i^j) \nabla_{E_j} A\psi + \frac{1}{4} \sum_{j,k=1}^n \tilde{\Gamma}_{ij}^k E_j \cdot E_k \cdot A\psi. \end{aligned} \quad (7)$$

Hence we obtain

$$\begin{aligned} AD^g \psi &= D^{g_{\text{eucl}}} A\psi + \sum_{i,j=1}^n (B_i^j - \delta_i^j) E_i \cdot \nabla_{E_j} A\psi \\ &\quad + \frac{1}{4} \sum_{i,j,k=1}^n \tilde{\Gamma}_{ij}^k E_i \cdot E_j \cdot E_k \cdot A\psi. \end{aligned} \quad (8)$$

Let  $\partial_j := d\rho(E_j)$ ,  $1 \leq j \leq n$ , be the coordinate vector fields of the normal coordinates. The Taylor expansion of the coefficient  $g_{ij}$  of the metric around 0 is given by

$$g_{ij}(x) = \delta_{ij} + \frac{1}{3} \sum_{a,b=1}^n R_{iabj}(p) x_a x_b + O(|x|^3),$$

where  $R_{iabj} = g(R(\partial_b, \partial_j) \partial_a, \partial_i)$  denotes the components of the Riemann curvature tensor (see e.g. [LP], p. 61). Since we have  $(B_i^j)_{ij} = (g_{ij})_{ij}^{-1/2}$  it follows that

$$B_i^j(x) = \delta_i^j - \frac{1}{6} \sum_{a,b=1}^n R_{iabj}(p) x_a x_b + O(|x|^3). \quad (9)$$



Since we have  $\nabla_{\partial_k}^g \partial_r|_p = 0$  for all  $k, r$ , we obtain

$$\tilde{\Gamma}_{kr}^m(\rho(x)) = O(|x|) \quad (10)$$

as  $x \rightarrow 0$  for all  $m, k, r$ .

### 3.2 The Euclidean Dirac operator

The aim of this section is to calculate preimages under the Dirac operator of certain spinors on  $\mathbb{R}^n \setminus \{0\}$  with the Euclidean metric. The results will be useful for obtaining the expansion of Green's function for the Dirac operator on a closed spin manifold.

**Definition 3.1.** For  $k \in \mathbb{R}$ ,  $m \in \mathbb{N}$  and  $i \in \{0, 1\}$  we define the vector subspaces  $P_{k,m,i}(\mathbb{R}^n)$  of  $C^\infty(\Sigma \mathbb{R}^n|_{\mathbb{R}^n \setminus \{0\}})$  as follows. For  $k \neq 0$  define

$$\begin{aligned} P_{k,m,0}(\mathbb{R}^n) &:= \text{span} \left\{ x \mapsto x_{i_1} \dots x_{i_m} |x|^k \gamma \mid \begin{array}{l} 1 \leq i_1, \dots, i_m \leq n, \\ \gamma \in \Sigma_n \text{ constant} \end{array} \right\} \\ P_{k,m,1}(\mathbb{R}^n) &:= \text{span} \left\{ x \mapsto x_{i_1} \dots x_{i_m} |x|^k x \cdot \gamma \mid \begin{array}{l} 1 \leq i_1, \dots, i_m \leq n, \\ \gamma \in \Sigma_n \text{ constant} \end{array} \right\} \end{aligned}$$

and furthermore define

$$\begin{aligned} P_{0,m,0}(\mathbb{R}^n) &:= \text{span} \left\{ x \mapsto x_{i_1} \dots x_{i_m} \ln |x| \gamma \mid \begin{array}{l} 1 \leq i_1, \dots, i_m \leq n, \\ \gamma \in \Sigma_n \text{ constant} \end{array} \right\} \\ P_{0,m,1}(\mathbb{R}^n) &:= \text{span} \left\{ x \mapsto x_{i_1} \dots x_{i_m} (1 - n \ln |x|) x \cdot \gamma \mid \begin{array}{l} 1 \leq i_1, \dots, i_m \leq n, \\ \gamma \in \Sigma_n \text{ constant} \end{array} \right\}. \end{aligned}$$

Note that there exist inclusions among these spaces. For example one has  $P_{k+2,m,0}(\mathbb{R}^n) \subset P_{k,m+2,0}(\mathbb{R}^n)$  for all  $m$  and all  $k \neq -2$ . However in the following we will very often not use these inclusions, since exceptions like the case  $k = -2$  in this example would make the following statements rather more complicated. The exception  $k = -2$  in this example is due to our definition of  $P_{0,m,0}(\mathbb{R}^n)$ . The following proposition will show that this definition is nevertheless useful.

**Proposition 3.2.** For all  $m \in \mathbb{N}$ ,  $k \in \mathbb{R}$  with  $-n \leq k$  and  $-n < k + m \leq 0$  we have

$$P_{k,m,0}(\mathbb{R}^n) \subset D^{\text{eucl}} \left( \sum_{j=1}^{[(m+1)/2]} P_{k+2j,m+1-2j,0}(\mathbb{R}^n) + \sum_{j=0}^{[m/2]} P_{k+2j,m-2j,1}(\mathbb{R}^n) \right)$$

For all  $m \in \mathbb{N}$ ,  $k \in \mathbb{R}$  with  $-n \leq k$  and  $-n < k + m + 1 \leq 0$  we have

$$P_{k,m,1}(\mathbb{R}^n) \subset D^{g_{\text{eucl}}} \left( \sum_{j=0}^{\lfloor m/2 \rfloor} P_{k+2+2j,m-2j,0}(\mathbb{R}^n) + \sum_{j=1}^{\lfloor (m+1)/2 \rfloor} P_{k+2j,m+1-2j,1}(\mathbb{R}^n) \right).$$

*Proof.* We use induction on  $m$ . Let  $m = 0$  and let  $\gamma$  be a constant spinor on  $(\mathbb{R}^n, g_{\text{eucl}})$ . We want to prove that  $P_{k,0,0}(\mathbb{R}^n) \subset D^{g_{\text{eucl}}}(P_{k,0,1}(\mathbb{R}^n))$  for all  $k$  with  $-n < k \leq 0$  and  $P_{k,0,1}(\mathbb{R}^n) \subset D^{g_{\text{eucl}}}(P_{k+2,0,0}(\mathbb{R}^n))$  for all  $k$  with  $-n \leq k$  and  $k + 1 \leq 0$ . One calculates easily that

$$\begin{aligned} D^{g_{\text{eucl}}} \left( -\frac{1}{n+k} |x|^k x \cdot \gamma \right) &= |x|^k \gamma, \quad k \neq -n \\ D^{g_{\text{eucl}}} \left( \frac{1-n \ln |x|}{n^2} x \cdot \gamma \right) &= \ln |x| \gamma, \\ D^{g_{\text{eucl}}} \left( \frac{1}{k+2} |x|^{k+2} \gamma \right) &= |x|^k x \cdot \gamma, \quad k \neq -2 \\ D^{g_{\text{eucl}}} (\ln |x| \gamma) &= |x|^{-2} x \cdot \gamma. \end{aligned}$$

Using the definition of  $P_{k,0,i}(\mathbb{R}^n)$  one finds that the assertion holds for  $m = 0$ .

Let  $m \geq 1$  and assume that all the inclusions in the assertion hold for  $m - 1$ . Using the equation  $E_i \cdot x = -2x_i - x \cdot E_i$  we find

$$\begin{aligned} & D^{g_{\text{eucl}}} (-x_{i_1} \dots x_{i_m} |x|^k x \cdot \gamma) \\ &= - \sum_{j=1}^m x_{i_1} \dots \widehat{x_{i_j}} \dots x_{i_m} |x|^k E_{i_j} \cdot x \cdot \gamma - x_{i_1} \dots x_{i_m} D^{g_{\text{eucl}}} (|x|^k x \cdot \gamma) \\ &= (2m + n + k) x_{i_1} \dots x_{i_m} |x|^k \gamma + \sum_{j=1}^m x_{i_1} \dots \widehat{x_{i_j}} \dots x_{i_m} |x|^k x \cdot E_{i_j} \cdot \gamma. \end{aligned}$$

Since  $E_{i_j} \cdot \gamma$  is a parallel spinor the sum on the right hand side is contained in  $P_{k,m-1,1}(\mathbb{R}^n)$ . We apply the induction hypothesis and since  $2m + n + k \neq 0$  we find that the assertion for  $P_{k,m,0}(\mathbb{R}^n)$  holds. We define  $f_k(x) := \frac{1}{k} |x|^k$  for  $k \neq 0$  and  $f_0(x) := \ln |x|$ . Then we find

$$\begin{aligned} & D^{g_{\text{eucl}}} (x_{i_1} \dots x_{i_m} f_{k+2}(x) \gamma) \\ &= x_{i_1} \dots x_{i_m} |x|^k x \cdot \gamma + \sum_{j=1}^m x_{i_1} \dots \widehat{x_{i_j}} \dots x_{i_m} f_{k+2}(x) E_{i_j} \cdot \gamma. \end{aligned}$$

The sum on the right hand side is in  $P_{k+2,m-1,0}(\mathbb{R}^n)$ . Again we apply the induction hypothesis and we find that the assertion for  $P_{k,m,1}(\mathbb{R}^n)$  holds.  $\square$

### 3.3 Expansion of Green's function

Let  $(M, g, \Theta)$  be a closed Riemannian spin manifold of dimension  $n$  and let  $\lambda \in \mathbb{R}$ . Let  $\pi_i: M \times M \rightarrow M$ ,  $i = 1, 2$  be the projections. We define

$$\Sigma^g M \boxtimes \Sigma^g M^* := \pi_1^* \Sigma^g M \otimes (\pi_2^* \Sigma^g M)^*$$

i. e.  $\Sigma^g M \boxtimes \Sigma^g M^*$  is the vector bundle over  $M \times M$  whose fibre over the point  $(x, y) \in M \times M$  is given by  $\text{Hom}_{\mathbb{C}}(\Sigma_y^g M, \Sigma_x^g M)$ . Let  $\Delta := \{(x, x) | x \in M\}$  be the diagonal. In the following we will abbreviate

$$\int_{M \setminus \{p\}} := \lim_{\varepsilon \rightarrow 0} \int_{M \setminus B_\varepsilon(p)}.$$

**Definition 3.3.** A smooth section  $G_\lambda^g: M \times M \setminus \Delta \rightarrow \Sigma^g M \boxtimes \Sigma^g M^*$  which is locally integrable on  $M \times M$  is called a Green's function for  $D^g - \lambda$  if for all  $p \in M$ , for all  $\varphi \in \Sigma_p^g M$  and for all  $\psi \in \text{im}(D^g - \lambda)$  we have

$$\int_{M \setminus \{p\}} \langle (D^g - \lambda)\psi, G_\lambda^g(., p)\varphi \rangle d\mathbf{v}^g = \langle \psi(p), \varphi \rangle, \quad (11)$$

and if for all  $p \in M$ , for all  $\varphi \in \Sigma_p^g M$  and for all  $\psi \in \ker(D^g - \lambda)$  we have

$$\int_{M \setminus \{p\}} \langle \psi, G_\lambda^g(., p)\varphi \rangle d\mathbf{v}^g = 0. \quad (12)$$

The smooth spinor  $G_\lambda^g(., p)\varphi$  on  $M \setminus \{p\}$  will sometimes also be called Green's function for  $\varphi$ . Let  $P: C^\infty(\Sigma^g M) \rightarrow \ker(D^g - \lambda)$  denote the  $L^2$ -orthogonal projection. Then we have for all  $\psi \in C^\infty(\Sigma^g M)$  and for all  $\varphi \in \Sigma_p^g M$

$$\int_{M \setminus \{p\}} \langle (D^g - \lambda)\psi, G_\lambda^g(., p)\varphi \rangle d\mathbf{v}^g = \langle \psi(p) - P\psi(p), \varphi \rangle. \quad (13)$$

On Euclidean space we define a Green's function as follows.

**Definition 3.4.** Let  $(M, g) = (\mathbb{R}^n, g_{\text{eucl}})$  with the unique spin structure. A smooth section  $G_\lambda^g: M \times M \setminus \Delta \rightarrow \Sigma^g M \boxtimes \Sigma^g M^*$  which is locally integrable on  $M \times M$  is called a Green's function for  $D^g - \lambda$  if for all  $p \in M$ , for all  $\varphi \in \Sigma_p^g M$  and for all  $\psi \in C^\infty(\Sigma^g M)$  with compact support the equation (11) holds.

Of course a Green's function for  $D^{g_{\text{eucl}}} - \lambda$  is not uniquely determined by this definition. We will explicitly write down a Green's function for  $D^{g_{\text{eucl}}} - \lambda$ . First observe that for every spinor  $\chi \in C^\infty(\Sigma \mathbb{R}^n|_{\mathbb{R}^n \setminus \{0\}})$  and for every  $\lambda \in \mathbb{R}$  the equation

$$(D^{g_{\text{eucl}}} - \lambda)(D^{g_{\text{eucl}}} + \lambda)\chi = - \sum_{i=1}^n \nabla_{E_i} \nabla_{E_i} \chi - \lambda^2 \chi$$

holds on  $\mathbb{R}^n \setminus \{0\}$ . Let  $\gamma$  be a constant spinor on  $(\mathbb{R}^n, g_{\text{eucl}})$  and let  $g$  be a solution to the ordinary differential equation

$$g''(z) + \frac{n-1}{z}g'(z) + \lambda^2 g(z) = -\delta_0, \quad (14)$$

which is smooth on  $(0, \infty)$ . If we define  $f \in C^\infty(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$  by  $f(x) := g(|x|)$ , then the spinor  $G_\lambda^{g_{\text{eucl}}}(\cdot, 0)\gamma := (D^{g_{\text{eucl}}} + \lambda)(f\gamma)$  is a Green's function.

In the following let  $\Gamma$  denote the Gamma function and  $J_m, Y_m$  the Bessel functions of the first and second kind for the parameter  $m \in \mathbb{R}$ . In the notation of [AS], p. 360 they are defined for  $z \in (0, \infty)$  by

$$\begin{aligned} J_m(z) &= \frac{1}{2^m \Gamma(m+1)} z^m \left(1 + \sum_{k=1}^{\infty} a_k z^{2k}\right), \quad m \in \mathbb{R}, \\ Y_0(z) &= \frac{2}{\pi} \left(\ln\left(\frac{z}{2}\right) + c\right) J_0(z) + \sum_{k=1}^{\infty} b_k z^{2k}, \\ Y_m(z) &= -\frac{2^m}{\pi} \Gamma(m) z^{-m} \left(1 + \sum_{k=1}^{\infty} c_k z^{2k}\right), \quad m = \frac{1}{2} + k, k \in \mathbb{N}, \\ Y_m(z) &= -\frac{2^m}{\pi} \Gamma(m) z^{-m} \left(1 + \sum_{k=1}^{\infty} d_k z^{2k}\right) + \frac{2}{\pi} \ln\left(\frac{z}{2}\right) J_m(z), \quad m \in \mathbb{N} \setminus \{0\}, \end{aligned}$$

where  $c$  is a real constant, the  $a_k, b_k, c_k, d_k$  are real coefficients, the  $a_k, c_k, d_k$  depend on  $m$  and all the power series converge for all  $z \in (0, \infty)$ . Let  $\omega_{n-1} = \text{vol}(S^{n-1}, g_{\text{can}})$  be the volume of the  $(n-1)$ -dimensional unit sphere with the standard metric.

**Theorem 3.5.** *Let  $m := \frac{n-2}{2}$ . We define  $f_\lambda: \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$  as follows. For  $\lambda \neq 0$  and  $n = 2$*

$$f_\lambda(x) := -\frac{1}{4} Y_0(|\lambda x|) + \frac{\ln|\lambda| - \ln(2) + c}{2\pi} J_0(|\lambda x|),$$

for  $\lambda \neq 0$  and odd  $n \geq 3$

$$f_\lambda(x) := -\frac{\pi|\lambda|^m}{2^m\Gamma(m)(n-2)\omega_{n-1}}|x|^{-m}Y_m(|\lambda x|),$$

for  $\lambda \neq 0$  and even  $n \geq 4$

$$f_\lambda(x) := -\frac{\pi|\lambda|^m}{2^m\Gamma(m)(n-2)\omega_{n-1}}|x|^{-m}\left(Y_m(|\lambda x|) - \frac{2(\ln|\lambda| - \ln(2))}{\pi}J_m(|\lambda x|)\right)$$

and

$$f_0(x) := -\frac{1}{2\pi}\ln|x|, \quad n=2, \quad f_0(x) := \frac{1}{(n-2)\omega_{n-1}|x|^{n-2}}, \quad n \geq 3.$$

Then for every constant spinor  $\gamma$  on  $\mathbb{R}^n$  a Green's function for  $D^{g_{\text{eucl}}} - \lambda$  is given by

$$G_\lambda^{g_{\text{eucl}}}(x, 0)\gamma = (D^{g_{\text{eucl}}} + \lambda)(f_\lambda\gamma)(x).$$

**Corollary 3.6.** For every constant spinor  $\gamma \in \Sigma_n$  there exists a Green's function of  $D^{g_{\text{eucl}}} - \lambda$ , which has the following form. For  $n = 2$

$$G_\lambda^{g_{\text{eucl}}}(x, 0)\gamma = -\frac{1}{2\pi|x|}\frac{x}{|x|} \cdot \gamma - \frac{\lambda}{2\pi}\ln|x|\gamma + \ln|x|\vartheta_\lambda(x) + \zeta_\lambda(x),$$

for odd  $n \geq 3$

$$G_\lambda^{g_{\text{eucl}}}(x, 0)\gamma = -\frac{1}{\omega_{n-1}|x|^{n-1}}\frac{x}{|x|} \cdot \gamma + \frac{\lambda}{(n-2)\omega_{n-1}|x|^{n-2}}\gamma + |x|^{2-n}\zeta_\lambda(x),$$

for even  $n \geq 4$

$$\begin{aligned} G_\lambda^{g_{\text{eucl}}}(x, 0)\gamma &= -\frac{1}{\omega_{n-1}|x|^{n-1}}\frac{x}{|x|} \cdot \gamma + \frac{\lambda}{(n-2)\omega_{n-1}|x|^{n-2}}\gamma + |x|^{2-n}\zeta_\lambda(x) \\ &\quad - \frac{\lambda^{n-1}}{2^{n-2}\Gamma(\frac{n}{2})^2\omega_{n-1}}\ln|x|\gamma + \ln|x|\vartheta_\lambda(x), \end{aligned}$$

where for every  $n$  and for every  $\lambda$  the spinors  $\vartheta_\lambda$ ,  $\zeta_\lambda$  extend smoothly to  $\mathbb{R}^n$  and satisfy

$$|\zeta_\lambda(x)|_{g_{\text{eucl}}} = O(|x|), \quad |\vartheta_\lambda(x)|_{g_{\text{eucl}}} = O(|x|) \quad \text{as } x \rightarrow 0$$

and where for every  $n$  and for every  $x$  the spinors  $\vartheta_\lambda(x), \zeta_\lambda(x) \in \Sigma_n$  are power series in  $\lambda$  with  $\vartheta_0(x) = \zeta_0(x) = 0$ .

*Proof of Corollary 3.6.* Using the definition of  $f_\lambda$  we find for  $n = 2$

$$f_\lambda(x) = -\frac{1}{2\pi} \ln |x| \left(1 + \sum_{k=1}^{\infty} a_k |\lambda x|^{2k}\right) - \frac{1}{4} \sum_{k=1}^{\infty} b_k |\lambda x|^{2k},$$

for odd  $n \geq 3$

$$f_\lambda(x) = \frac{1}{(n-2)\omega_{n-1}|x|^{n-2}} \left(1 + \sum_{k=1}^{\infty} c_k |\lambda x|^{2k}\right)$$

and for even  $n \geq 4$

$$\begin{aligned} f_\lambda(x) &= \frac{1}{(n-2)\omega_{n-1}|x|^{n-2}} \left(1 + \sum_{k=1}^{\infty} d_k |\lambda x|^{2k}\right) \\ &\quad - \frac{\lambda^{n-2}}{2^{n-2}(m!)^2\omega_{n-1}} \ln |x| \left(1 + \sum_{k=1}^{\infty} a_k |\lambda x|^{2k}\right). \end{aligned}$$

The assertion follows.  $\square$

*Proof of Theorem 3.5.* Let  $f_\lambda$  be as in the assertion and write  $f_\lambda(x) = g_\lambda(|x|)$  with  $g_\lambda: (0, \infty) \rightarrow \mathbb{R}$ . One finds that  $g_\lambda$  solves the equation (14). It remains to show that  $(D^{g_{\text{eucl}}} + \lambda)(f_\lambda \gamma)$  satisfies (11). The calculation of the proof of the Corollary shows

$$(D^{g_{\text{eucl}}} + \lambda)(f_\lambda \gamma)(x) = -\frac{1}{\omega_{n-1}} \frac{x}{|x|^n} \cdot \gamma + \zeta(x),$$

where  $|\zeta(x)|_{g_{\text{eucl}}} = o(|x|^{1-n})$  as  $x \rightarrow 0$ . For any Riemannian spin manifold  $(M, g, \Theta)$  with boundary  $\partial M$  and  $\psi, \varphi$  compactly supported spinors we have

$$(D^g \psi, \varphi)_2 - (\psi, D^g \varphi)_2 = \int_{\partial M} \langle \nu \cdot \psi, \varphi \rangle dA,$$

where  $\nu$  is the outer unit normal vector field on  $\partial M$  (see [LM], p. 115). We apply this to  $(\mathbb{R}^n \setminus B_\varepsilon(0), g_{\text{eucl}})$  and  $\nu(x) := -\frac{x}{|x|}$  and we obtain the assertion.  $\square$

**Definition 3.7.** For  $m \in \mathbb{R}$  we define

$$P_m(\mathbb{R}^n) := \sum_{\substack{r+s+t \geq m \\ r \geq -n}} P_{r,s,t}(\mathbb{R}^n) + (C^\infty(\Sigma \mathbb{R}^n|_{\mathbb{R}^n \setminus \{0\}}) \cap C^0(\Sigma \mathbb{R}^n)),$$

where the second space is the space of all spinors which are smooth on  $\mathbb{R}^n \setminus \{0\}$  and have a continuous extension to  $\mathbb{R}^n$ .

**Remark 3.8.** Let  $\vartheta \in P_m(\mathbb{R}^n)$ . Then we have  $E_i \cdot \vartheta \in P_m(\mathbb{R}^n)$  for all  $i \in \{1, \dots, n\}$ . If  $f \in \mathbb{C}^\infty(\mathbb{R}^n)$  then using Taylor's formula for  $f$  we find that the spinor  $f\vartheta$  is in  $P_m(\mathbb{R}^n)$ .

**Remark 3.9.** A spinor  $\vartheta \in P_m(\mathbb{R}^n)$  has a continuous extension to  $\mathbb{R}^n$  if and only if  $m > 0$ . Furthermore by Proposition 3.2 it follows that for all  $m \in (-n, 0]$  we have

$$\begin{aligned} P_m(\mathbb{R}^n) &= \sum_{\substack{r+s+t=m \\ r \geq -n}} P_{r,s,t}(\mathbb{R}^n) + P_{m+1}(\mathbb{R}^n) \\ &\subset D^{g_{\text{eucl}}}(P_{m+1}(\mathbb{R}^n)) + P_{m+1}(\mathbb{R}^n). \end{aligned}$$

**Lemma 3.10.** Let  $(M, g, \Theta)$  be a closed Riemannian spin manifold of dimension  $n$  and let  $\lambda \in \mathbb{R}$ . Let  $\gamma \in \Sigma_n$  be a constant spinor on  $\mathbb{R}^n$ . Then the spinor  $G_\lambda^{g_{\text{eucl}}}(\cdot, 0)\gamma$  is in  $P_{1-n}(\mathbb{R}^n)$ . Let the matrix coefficients  $B_i^j$  be defined as in (8). Then for all  $i$  the spinor

$$x \mapsto \sum_{j=1}^n (B_i^j(x) - \delta_i^j) \nabla_{E_j} G_\lambda^{g_{\text{eucl}}}(x, 0)\gamma$$

is in  $P_{2-n}(\mathbb{R}^n)$ .

*Proof.* The assertion for  $G_\lambda^{g_{\text{eucl}}}(\cdot, 0)\gamma$  can be seen immediately from Corollary 3.6. Let  $f_\lambda$  be as in Theorem 3.5 and write  $f_\lambda(x) = g_\lambda(|x|)$  with a function  $g_\lambda: (0, \infty) \rightarrow \mathbb{R}$ . Then we have

$$G_\lambda^{g_{\text{eucl}}}(x, 0)\gamma = \frac{g'_\lambda(|x|)}{|x|} x \cdot \gamma + \lambda g_\lambda(|x|)\gamma$$

and for every  $j \in \{1, \dots, n\}$  we have

$$\nabla_{E_j} G_\lambda^{g_{\text{eucl}}}(x, 0)\gamma = \frac{g''_\lambda(|x|)x_j}{|x|^2} x \cdot \gamma - \frac{g'_\lambda(|x|)x_j}{|x|^3} x \cdot \gamma + \frac{g'_\lambda(|x|)}{|x|} E_j \cdot \gamma + \lambda \frac{g'_\lambda(|x|)x_j}{|x|} \gamma.$$

Since the exponential map is a radial isometry, we have  $\sum_{j=1}^n g_{ij}(x)x_j = x_i$  and thus  $\sum_{j=1}^n B_i^j(x)x_j = x_i$  for every fixed  $i$ . Thus we find

$$\sum_{j=1}^n (B_i^j(x) - \delta_i^j) \nabla_{E_j} G_\lambda^{g_{\text{eucl}}}(x, 0)\gamma = \sum_{j=1}^n (B_i^j(x) - \delta_i^j) \frac{g'_\lambda(|x|)}{|x|} E_j \cdot \gamma.$$

Since we have  $g'_\lambda(|x|) = O(|x|^{1-n})$  as  $x \rightarrow 0$  the assertion now follows from the Taylor expansion (9) of  $B_i^j(x)$ .  $\square$

Next we prove existence and uniqueness of Green's function for  $D^g - \lambda$  on a closed Riemannian spin manifold in such a way that we also obtain the expansion of Green's function around the singularity. The idea is to apply the equation (8) for the Dirac operator in the trivialization to a Euclidean Green's function and then determine the correction terms. This has been carried out in [AH], where for some technical steps Sobolev embeddings were used. We present a more simple argument using the preimages under the Dirac operator from Proposition 3.2.

In the following for a fixed point  $p \in M$  let  $\rho: V \rightarrow U$  be a local parametrization of  $M$  by Riemannian normal coordinates, where  $U \subset M$  is an open neighborhood of  $p$ ,  $V \subset \mathbb{R}^n$  is an open neighborhood of 0 and  $\rho(0) = p$ . Furthermore let

$$\beta: \Sigma \mathbb{R}^n|_V \rightarrow \Sigma^g M|_U, \quad A: C^\infty(\Sigma^g M|_U) \rightarrow C^\infty(\Sigma \mathbb{R}^n|_V)$$

denote the maps which send a spinor to its corresponding spinor in the Bourguignon-Gauduchon trivialization defined in Section 3.1.

**Theorem 3.11.** *Let  $(M, g, \Theta)$  be a closed  $n$ -dimensional Riemannian spin manifold,  $p \in M$ . For every  $\varphi \in \Sigma_p^g M$  there exists a unique Green's function  $G_\lambda^g(., p)\varphi$ . If  $\gamma := \beta^{-1}\varphi \in \Sigma_n$  is the constant spinor on  $\mathbb{R}^n$  corresponding to  $\varphi$ , then the first two terms of the expansion of  $AG_\lambda^g(., p)\varphi$  at 0 coincide with the first two terms of the expansion of  $G_\lambda^{g_{\text{eucl}}}(., 0)\gamma$  given in Corollary 3.6.*

*Proof.* Let  $\varepsilon > 0$  such that  $B_{2\varepsilon}(0) \subset V$  and let  $\eta: \mathbb{R}^n \rightarrow [0, 1]$  be a smooth function with  $\text{supp}(\eta) \subset B_{2\varepsilon}(0)$  and  $\eta \equiv 1$  on  $B_\varepsilon(0)$ . Then the spinor  $\Theta_1$  defined on  $\mathbb{R}^n \setminus \{0\}$  by  $\Theta_1(x) := \eta(x)G_\lambda^{g_{\text{eucl}}}(x, 0)\gamma$  is smooth on  $\mathbb{R}^n \setminus \{0\}$ . For  $r \in \{1, \dots, n\}$  we define smooth spinors  $\Phi_r$  on  $M \setminus \{p\}$  and  $\Theta_{r+1}$  on  $\mathbb{R}^n \setminus \{0\}$  inductively as follows. For  $r = 1$  we define

$$\Phi_1(q) := \begin{cases} A^{-1}\Theta_1(q), & q \in U \setminus \{p\} \\ 0, & q \in M \setminus U \end{cases}$$

and

$$\Theta_2(x) := \begin{cases} A(D^g - \lambda)\Phi_1(x), & x \in V \setminus \{0\} \\ 0, & x \in \mathbb{R}^n \setminus V \end{cases}.$$

By the formula (8) for the Dirac operator in the trivialization we have on  $V \setminus \{0\}$

$$\begin{aligned} \Theta_2 &= (D^{g_{\text{eucl}}} - \lambda)\Theta_1 + \sum_{i,j=1}^n (B_i^j - \delta_i^j)E_i \cdot \nabla_{E_j} \Theta_1 \\ &\quad + \frac{1}{4} \sum_{i,j,k=1}^n \tilde{\Gamma}_{ij}^k E_i \cdot E_j \cdot E_k \cdot \Theta_1. \end{aligned}$$



The first term vanishes on  $B_\varepsilon(0) \setminus \{0\}$ . It follows from the expansions of  $\tilde{\Gamma}_{ij}^k$  and  $B_i^j - \delta_i^j$  in (9), (10) and from Lemma 3.10 that  $\Theta_2 \in P_{2-n}(\mathbb{R}^n)$ .

Next let  $r \in \{2, \dots, n\}$  and assume that  $\Phi_{r-1}$  and  $\Theta_r$  have already been defined. We may assume that  $\Theta_r \in P_{r-n}(\mathbb{R}^n)$ . By Remark 3.9 there exists  $\beta_{r+1} \in P_{r+1-n}(\mathbb{R}^n)$  such that  $\Theta_r - (D^{g_{\text{eucl}}} - \lambda)\beta_{r+1} \in P_{r+1-n}(\mathbb{R}^n)$ . We define  $\Phi_r$  and  $\Theta_{r+1}$  by

$$\Phi_r(q) := \begin{cases} \Phi_{r-1}(q) - A^{-1}(\eta\beta_{r+1})(q), & q \in U \setminus \{p\} \\ 0, & q \in M \setminus U \end{cases}$$

and

$$\Theta_{r+1}(x) := \begin{cases} A(D^g - \lambda)\Phi_r(x), & x \in V \setminus \{0\} \\ 0, & x \in \mathbb{R}^n \setminus V \end{cases}.$$

By the formula (8) for the Dirac operator in the trivialization we have on  $B_\varepsilon(0) \setminus \{0\}$

$$\begin{aligned} \Theta_{r+1} &= A(D^g - \lambda)\Phi_{r-1} - A(D^g - \lambda)A^{-1}\beta_{r+1} \\ &= \Theta_r - (D^{g_{\text{eucl}}} - \lambda)\beta_{r+1} - \sum_{i,j=1}^n (B_i^j - \delta_i^j)E_i \cdot \nabla_{E_j}\beta_{r+1} \\ &\quad - \frac{1}{4} \sum_{i,j,k=1}^n \tilde{\Gamma}_{ij}^k E_i \cdot E_j \cdot E_k \cdot \beta_{r+1}. \end{aligned}$$

Using the expansions of  $\tilde{\Gamma}_{ij}^k$  and  $B_i^j - \delta_i^j$  in (9), (10) we conclude that  $\Theta_{r+1} \in P_{r+1-n}(\mathbb{R}^n)$ .

We see that  $\Theta_{n+1}$  has a continuous extension to  $\mathbb{R}^n$  and we obtain a continuous extension  $\Psi$  of  $(D^g - \lambda)\Phi_n$  to all of  $M$ . Thus there exists

$$\Psi' \in C^\infty(\Sigma^g M|_{M \setminus \{p\}}) \cap H^1(\Sigma^g M)$$

such that  $(D^g - \lambda)\Psi' = P\Psi - \Psi$ . Define

$$\Gamma := \Phi_n + \Psi', \quad \Theta := -\eta\beta_3 - \dots - \eta\beta_{n+1} + A\Psi'.$$

Then on  $B_\varepsilon(0) \setminus \{0\}$  we have  $A\Gamma = G_\lambda^{g_{\text{eucl}}}(\cdot, 0)\gamma + \Theta$ .

If  $P$  is the  $L^2$ -orthogonal projection onto  $\ker(D^g - \lambda)$ , then

$$G_\lambda^g(\cdot, p)\varphi := \Gamma - P\Gamma$$

satisfies (11), (12) and thus is a Green's function. Uniqueness also follows from (11), (12). The statement on the expansion of  $AG_\lambda^g(\cdot, p)\varphi$  is obvious, since we have  $\Theta \in P_{3-n}(\mathbb{R}^n)$ .  $\square$

## 4 Zero sets of eigenspinors

### 4.1 Eigenspinors in dimensions 2 and 3

In this section we prove our main result Theorem 1.1. Assume that  $n \in \{2, 3\}$ . Then there exists a quaternionic structure on the spinor bundle, i. e. a conjugate linear endomorphism  $J$  of  $\Sigma^g M$ , which satisfies  $J^2 = -\text{Id}$ . Furthermore  $J$  is parallel and commutes with Clifford multiplication. It follows that  $J$  commutes with the Dirac operator and thus every eigenspace of  $D^g$  has even complex dimension. Therefore the following notation introduced in [Da] is useful.

**Definition 4.1.** *Let  $n \in \{2, 3\}$ . An eigenvalue  $\lambda$  of  $D^g$  is called simple, if one has  $\dim_{\mathbb{C}} \ker(D^g - \lambda) = 2$ .*

In this case one can choose an  $L^2$ -orthonormal basis of  $\ker(D^g - \lambda)$  of the form  $\{\psi, J\psi\}$ .

**Definition 4.2.** *A subset of  $R(M)$  is called generic, if it is open in  $R(M)$  with respect to the  $C^1$ -topology and dense in  $R(M)$  with respect to all  $C^k$ -topologies,  $k \in \mathbb{N}$ .*

**Remark 4.3.** *Let  $k \in \mathbb{N}$  and let  $A \subset R(M)$ . If  $A$  is open in  $R(M)$  with respect to the  $C^k$ -topology, then it is open in  $R(M)$  with respect to the  $C^m$ -topology for all  $m > k$ . If  $A$  is dense in  $R(M)$  with respect to the  $C^k$ -topology, then it is dense in  $R(M)$  with respect to the  $C^m$ -topology for all  $m < k$ .*

For every  $g \in R(M)$  we enumerate the nonzero eigenvalues of  $D^g$  in the following way

$$\dots \leq \lambda_2^- \leq \lambda_1^- < 0 < \lambda_1^+ \leq \lambda_2^+ \leq \dots$$

Here all the non-zero eigenvalues are repeated by their complex multiplicities, while  $\dim \ker(D^g) \geq 0$  is arbitrary. For  $m \in \mathbb{N} \setminus \{0\}$  we define

$$\begin{aligned} S_m(M) &:= \{g \in R(M) \mid \lambda_1^\pm, \dots, \lambda_m^\pm \text{ are simple}\} \\ N_m(M) &:= \{g \in R(M) \mid \text{all eigenspinors to } \lambda_1^\pm, \dots, \lambda_m^\pm \text{ are nowhere zero}\}. \end{aligned}$$

It has been shown by M. Dahl that for every  $m \in \mathbb{N}$  the subset  $S_m(M) \cap [g]$  is generic in  $[g]$  (see [Da]). We are going to use this result later. In order to prove Theorem 1.1 it is sufficient to prove the following theorem.

**Theorem 4.4.** *Let  $M$  be a closed connected spin manifold of dimension 2 or 3 and let  $m \in \mathbb{N} \setminus \{0\}$ . Then for every  $g \in R(M)$  the subset  $N_m(M) \cap [g]$  is generic in  $[g]$ .*

In order to show density we will use Theorem 2.3 for families of spinors parametrized by Riemannian metrics. A basic problem is that  $[g]$  is not a smooth manifold. If one replaces  $[g]$  by the space of all  $k$  times continuously differentiable metrics,  $1 \leq k < \infty$ , which are conformal to  $g$ , then the coefficients of the Dirac operator are not smooth in general. In this case we cannot expect that the eigenspinors are smooth and that Theorem 2.4 remains valid. In order to get around this problem we will use finite dimensional manifolds of smooth Riemannian metrics of the form

$$V_{f_1 \dots f_r} := \left\{ \left( 1 + \sum_{i=1}^r t_i f_i \right) g \mid t_1, \dots, t_r \in \mathbb{R} \right\} \cap R(M),$$

where  $r \in \mathbb{N} \setminus \{0\}$  and  $f_1, \dots, f_r \in C^\infty(M, \mathbb{R})$ .

Our first aim is to construct a map, which associates to a Riemannian metric  $h \in [g]$  an eigenspinor of  $D^{g,h}$  in a continuous way.

**Lemma 4.5.** *Let  $k, m \in \mathbb{N} \setminus \{0\}$ . Let  $g \in S_m(M)$  and equip  $[g]$  with the  $C^k$ -topology. Let  $\lambda \in \{\lambda_1^\pm(g), \dots, \lambda_m^\pm(g)\}$  and let  $\psi$  be an eigenspinor of  $D^g$  corresponding to  $\lambda$ . Then there exists an open neighborhood  $V \subset [g]$  of  $g$  and a map  $F_\psi: V \rightarrow C^\infty(\Sigma^g M)$  such that for every  $h \in V$  the spinor  $F_\psi(h)$  is an eigenspinor of  $D^{g,h}$  and such that the map  $F_\psi^{ev}: V \times M \rightarrow \Sigma^g M$  defined by  $F_\psi^{ev}(h, x) := F_\psi(h)(x)$  is continuous and such that for all functions  $f_1, \dots, f_r \in C^\infty(M, \mathbb{R})$  the restriction  $F_\psi^{ev}|_{V_{f_1 \dots f_r} \times M}$  is differentiable.*

*Proof.* Let  $V \subset [g]$  be an open convex neighborhood of  $g$ , which is contained in  $S_m(M) \cap [g]$ . Let  $h \in V$  and let  $I \subset \mathbb{R}$  be an open interval containing  $[0, 1]$  such that for every  $t \in I$  the tensor field  $g_t := g + t(h - g)$  is a Riemannian metric on  $M$ . Let  $\lambda \in \{\lambda_1^\pm(g), \dots, \lambda_m^\pm(g)\}$ . By Lemma 2.1 there exists exactly one real analytic family  $t \mapsto \lambda_t$  of eigenvalues of  $D^{g, g_t}$  such that  $\lambda_0 = \lambda$ . Furthermore there exists a real analytic family  $\psi_t$  of spinors, such that  $\{\psi_t, J\psi_t\}$  forms an  $L^2$ -orthonormal basis of  $\ker(D^{g, g_t} - \lambda_t)$  for every  $t$ . After possibly shrinking  $V$  we may assume that for all metrics  $h \in V$  and for all  $\lambda \in \{\lambda_1^\pm(g), \dots, \lambda_m^\pm(g)\}$  the families  $\lambda_t$  and  $\psi_t$  are defined for all  $t \in [0, 1]$ . Since we are free to replace  $\psi_t, J\psi_t$  by linear combinations  $a\psi_t + bJ\psi_t$  with  $a, b \in \mathbb{C}, |a|^2 + |b|^2 > 0$ , we may assume that  $\psi_0 = \psi$ . We define  $F_\psi(h) := \psi_1$ .

Let  $h \in V$  and let  $k = fg$  for some  $f \in C^\infty(M, \mathbb{R})$ . With  $h_t := h + tk$  consider  $F_\psi(h_t)$  for small  $|t|$  such that  $h_t \in V$ . From (3) we obtain

$$D^{g, h_t} = \bar{\beta}_{h, g} \circ D^{h, h_t} \circ \bar{\beta}_{g, h}.$$

There exists a real analytic family  $t \mapsto \chi(t) \in C^\infty(\Sigma^h M)$  of eigenspinors of  $D^{h, h_t}$  with  $\chi(0) = \bar{\beta}_{g, h}(F_\psi(h))$ . It follows that

$$F_\psi(h_t) = \bar{\beta}_{h, g}(\chi(t)).$$

By taking  $k := h - h'$  for  $h, h' \in V$  we find that  $F_\psi^{ev}$  is continuous. Furthermore for all functions  $f_1, \dots, f_r \in C^\infty(M, \mathbb{R})$  we can use this equation to see that the restriction  $F_\psi^{ev}|_{V_{f_1 \dots f_r} \times M}$  is differentiable.  $\square$

The strategy for the proof of Theorem 4.4 is based on the following remark.

**Remark 4.6.** *Let  $A \subset \Sigma^g M$  be the zero section. Since the dimension of the total space  $\Sigma^g M$  of the spinor bundle is*

$$\dim \Sigma^g M = n + 2^{1+[n/2]} > 2n = \dim M + \dim A,$$

*a map  $f: M \rightarrow \Sigma^g M$  is transverse to  $A$  if and only if  $f^{-1}(A) = \emptyset$ .*

If for every  $\lambda_i \in \{\lambda_1^\pm, \dots, \lambda_m^\pm\}$  we choose an eigenspinor  $\psi_i$  and define the map

$$F_{\psi_i}: V_{f_1 \dots f_r} \rightarrow C^\infty(\Sigma^g M)$$

as in Lemma 4.5, then by this remark  $V_{f_1 \dots f_r} \cap N_m(M)$  is the set of all Riemannian metrics  $h \in V_{f_1 \dots f_r}$  such that all the  $F_{\psi_i}(h)$  are transverse to the zero section. Therefore in order to prove that  $N_m(M) \cap [g]$  is dense in  $[g]$  we would like to apply Theorem 2.3. Our aim is then to show that a suitable restriction of the map  $F_\psi^{ev}$  defined as in Lemma 4.5 is transverse to the zero section.

Let  $p \in M$  with  $\psi(p) = 0$ . We have a canonical decomposition of the tangent space

$$T_{\psi(p)} \Sigma^g M \cong \Sigma_p^g M \oplus T_p M$$

and thus

$$dF_\psi^{ev}|_{(g,p)}: T_g V_{f_1 \dots f_r} \oplus T_p M \rightarrow \Sigma_p^g M \oplus T_p M.$$

For a given  $h \in V_{f_1 \dots f_r}$  we will write  $g_t = g + t(h - g)$  and

$$\psi_t := F_\psi(g_t), \quad \frac{d\psi_t(x)}{dt}\Big|_{t=0} := \pi_1(dF_\psi^{ev}|_{(g,x)}(h - g, 0)).$$

Then it follows that

$$0 = \left(\frac{d}{dt} D^{g, g_t}\Big|_{t=0} - \frac{d\lambda_t}{dt}\Big|_{t=0}\right) \psi + (D^g - \lambda) \frac{d\psi_t}{dt}\Big|_{t=0}. \quad (15)$$

**Remark 4.7.** *Let  $\varphi \in \Sigma_p^g M$  and  $X, Y \in T_p M$ . If one polarizes the identity*

$$\langle X \cdot \varphi, X \cdot \varphi \rangle = g(X, X) \langle \varphi, \varphi \rangle,$$

then one obtains

$$\operatorname{Re}\langle X \cdot \varphi, Y \cdot \varphi \rangle = g(X, Y)\langle \varphi, \varphi \rangle. \quad (16)$$

Since Clifford multiplication with vectors is antisymmetric, it follows that  $\operatorname{Re}\langle X \cdot \varphi, \varphi \rangle = 0$ . Let  $\varphi \neq 0$  and let  $(e_i)_{i=1}^n$  be an orthonormal basis of  $T_p M$ . It follows that for  $n = 2$  the spinors

$$\varphi, e_1 \cdot \varphi, e_2 \cdot \varphi, e_1 \cdot e_2 \cdot \varphi$$

form an orthogonal basis of  $\Sigma_p^g M$  with respect to the real scalar product  $\operatorname{Re}\langle \cdot, \cdot \rangle$ . Similarly for  $n = 3$  the spinors

$$\varphi, e_1 \cdot \varphi, e_2 \cdot \varphi, e_3 \cdot \varphi$$

form an orthogonal basis of  $\Sigma_p^g M$  with respect to  $\operatorname{Re}\langle \cdot, \cdot \rangle$ .

The following rather long lemma is the most important step in showing that a suitable restriction of  $F_\psi^{ev}$  is transverse to the zero section.

**Lemma 4.8.** *Let  $n \in \{2, 3\}$  and  $m \in \mathbb{N} \setminus \{0\}$ . Let  $g \in S_m(M)$  and let  $\lambda \in \{\lambda_1^\pm(g), \dots, \lambda_m^\pm(g)\}$ . Let  $\psi$  be an eigenspinor of  $D^g$  corresponding to  $\lambda$  and let  $p \in M$  with  $\psi(p) = 0$ . Then there exist  $f_1, \dots, f_4 \in C^\infty(M, \mathbb{R})$  such that the map  $F_\psi^{ev}: V_{f_1 \dots f_4} \times M \rightarrow \Sigma^g M$  satisfies*

$$\pi_1(dF_\psi^{ev}|_{(g,p)}(T_g V_{f_1 \dots f_4} \oplus \{0\})) = \Sigma_p^g M.$$

*Proof.* Assume that the claim is wrong. Then there exists  $\varphi \in \Sigma_p^g M \setminus \{0\}$  such that for all  $f \in C^\infty(M, \mathbb{R})$  we have

$$0 = \operatorname{Re}\langle \pi_1(dF_\psi^{ev}|_{(g,p)}(fg, 0)), \varphi \rangle = \operatorname{Re}\langle \frac{d\psi_t}{dt}|_{t=0}(p), \varphi \rangle.$$

From the formula (13) for Green's function it follows that

$$0 = \operatorname{Re} \int_{M \setminus \{p\}} \langle (D^g - \lambda) \frac{d\psi_t}{dt}|_{t=0}, G_\lambda^g(\cdot, p) \varphi \rangle dv^g + \operatorname{Re} \langle P(\frac{d\psi_t}{dt}|_{t=0})(p), \varphi \rangle.$$

Since  $\lambda$  is a simple eigenvalue, all spinors in  $\ker(D^g - \lambda)$  vanish at  $p$ . Thus the last term vanishes. By (12) and (15) we have

$$\begin{aligned} 0 &= -\operatorname{Re} \int_{M \setminus \{p\}} \langle (\frac{d}{dt} D^{g, g_t}|_{t=0} - \frac{d\lambda_t}{dt}|_{t=0}) \psi, G_\lambda^g(\cdot, p) \varphi \rangle dv^g \\ &= -\operatorname{Re} \int_{M \setminus \{p\}} \langle \frac{d}{dt} D^{g, g_t}|_{t=0} \psi, G_\lambda^g(\cdot, p) \varphi \rangle dv^g \end{aligned}$$

for all  $f \in C^\infty(M, \mathbb{R})$ . If we use the formula (4) for the derivative of the Dirac operator and

$$\text{grad}^g(f) \cdot \psi = (D^g - \lambda)(f\psi)$$

it follows that

$$\begin{aligned} 0 &= \frac{1}{2} \text{Re} \int_{M \setminus \{p\}} \lambda f \langle \psi, G_\lambda^g(., p) \varphi \rangle \text{d}v^g \\ &\quad + \frac{1}{4} \text{Re} \int_{M \setminus \{p\}} \langle (D^g - \lambda)(f\psi), G_\lambda^g(., p) \varphi \rangle \text{d}v^g. \end{aligned}$$

Using the definition of Green's function and that all spinors in  $\ker(D^g - \lambda)$  vanish at  $p$ , we find that

$$\begin{aligned} 0 &= \frac{1}{2} \text{Re} \int_{M \setminus \{p\}} \lambda f \langle \psi, G_\lambda^g(., p) \varphi \rangle \text{d}v^g + \frac{1}{4} \text{Re} \langle (f\psi)(p) - P(f\psi)(p), \varphi \rangle \\ &= \frac{1}{2} \text{Re} \int_{M \setminus \{p\}} \lambda f \langle \psi, G_\lambda^g(., p) \varphi \rangle \text{d}v^g \end{aligned}$$

for all  $f \in C^\infty(M, \mathbb{R})$ . Since we have  $\lambda \neq 0$  it follows that  $\text{Re} \langle \psi, G_\lambda^g(., p) \varphi \rangle$  vanishes identically on  $M \setminus \{p\}$ .

Our aim is now to conclude that all the derivatives of  $\psi$  at the point  $p$  vanish. Then by Theorem 2.4 it follows that  $\psi$  is identically zero, which is a contradiction. In order to show this we choose a local parametrization  $\rho: V \rightarrow U$  of  $M$  by Riemannian normal coordinates, where  $U \subset M$  is an open neighborhood of  $p$ ,  $V \subset \mathbb{R}^n$  is an open neighborhood of 0 and  $\rho(0) = p$ . Furthermore let

$$\beta: \Sigma \mathbb{R}^n|_V \rightarrow \Sigma^g M|_U, \quad A: C^\infty(\Sigma^g M|_U) \rightarrow C^\infty(\Sigma \mathbb{R}^n|_V)$$

denote the maps which send a spinor to its corresponding spinor in the Bourguignon-Gauduchon trivialization defined in Section 3.1. We show by induction that  $\nabla^r A\psi(0) = 0$  for all  $r \in \mathbb{N}$ , where  $\nabla$  denotes the covariant derivative on  $\Sigma \mathbb{R}^n$ . The case  $r = 0$  is clear.

Let  $r \geq 1$  and assume that we have  $\nabla^s A\psi(0) = 0$  for all  $s \leq r - 1$ . Let  $(E_i)_{i=1}^n$  be the standard basis of  $\mathbb{R}^n$ . First consider the case  $n = 2$ . In the Bourguignon-Gauduchon trivialization we have

$$A\psi(x) = \frac{1}{r!} \sum_{j_1, \dots, j_r=1}^2 x_{j_1} \dots x_{j_r} \nabla_{E_{j_1}} \dots \nabla_{E_{j_r}} A\psi(0) + O(|x|^{r+1})$$

by Taylor's formula and

$$A(G_\lambda^g(., p)\varphi)(x) = -\frac{1}{2\pi|x|^2}x \cdot \gamma - \frac{\lambda}{2\pi} \ln|x|\gamma + O(|x|^0)$$

by Theorem 3.11, where  $\gamma := \beta^{-1}\varphi \in \Sigma_n$  is the constant spinor on  $\mathbb{R}^n$  corresponding to  $\varphi$ . It follows that

$$\begin{aligned} 0 &= -2\pi r!|x|^2 \text{Re}\langle A(G_\lambda^g(., p)\varphi)(x), A\psi(x) \rangle \\ &= \sum_{i,j_1,\dots,j_r=1}^2 x_i x_{j_1} \dots x_{j_r} \text{Re}\langle E_i \cdot \gamma, \nabla_{E_{j_1}} \dots \nabla_{E_{j_r}} A\psi(0) \rangle \\ &\quad + \lambda \sum_{j_1,\dots,j_r=1}^2 x_{j_1} \dots x_{j_r} |x|^2 \ln|x| \text{Re}\langle \gamma, \nabla_{E_{j_1}} \dots \nabla_{E_{j_r}} A\psi(0) \rangle + O(|x|^{r+2}) \end{aligned}$$

and therefore

$$\begin{aligned} 0 &= \text{Re}\langle E_i \cdot \gamma, \nabla_{E_{j_1}} \dots \nabla_{E_{j_r}} A\psi(0) \rangle \\ &\quad + \sum_{s=1}^r \text{Re}\langle E_{j_s} \cdot \gamma, \nabla_{E_i} \nabla_{E_{j_1}} \dots \widehat{\nabla_{E_{j_s}}} \dots \nabla_{E_{j_r}} A\psi(0) \rangle \end{aligned} \quad (17)$$

$$0 = \text{Re}\langle \gamma, \nabla_{E_{j_1}} \dots \nabla_{E_{j_r}} A\psi(0) \rangle \quad (18)$$

for all  $j_1, \dots, j_r, i \in \{1, 2\}$ , where the hat means that the operator is left out. By (18) and Remark 4.7 there exist  $a_{j_1, \dots, j_r, k}, b_{j_1, \dots, j_r} \in \mathbb{R}$  such that

$$\nabla_{E_{j_1}} \dots \nabla_{E_{j_r}} A\psi(0) = \sum_{k=1}^2 a_{j_1, \dots, j_r, k} E_k \cdot \gamma + b_{j_1, \dots, j_r} E_1 \cdot E_2 \cdot \gamma.$$

Observe that the coefficients  $a_{j_1, \dots, j_r, k}$  are symmetric in the first  $r$  indices. We insert this into (17) and we obtain

$$0 = a_{j_1, \dots, j_r, i} + \sum_{k=1}^r a_{i, j_1, \dots, \widehat{j_k}, \dots, j_r, j_k} \quad (19)$$

for all  $j_1, \dots, j_r, i \in \{1, 2\}$ . On the other hand since  $\psi \in \ker(D^g - \lambda)$  we find

using the induction hypothesis

$$\begin{aligned}
0 &= \lambda \nabla_{E_{j_1}} \dots \nabla_{E_{j_{r-1}}} A\psi(0) \\
&= \nabla_{E_{j_1}} \dots \nabla_{E_{j_{r-1}}} \sum_{i=1}^2 E_i \cdot \nabla_{E_i} A\psi(0) \\
&= \sum_{i,k=1}^2 a_{j_1, \dots, j_{r-1}, i, k} E_i \cdot E_k \cdot \gamma + \sum_{i=1}^2 b_{j_1, \dots, j_{r-1}, i} E_i \cdot E_1 \cdot E_2 \cdot \gamma \\
&= -(a_{j_1, \dots, j_{r-1}, 1, 1} + a_{j_1, \dots, j_{r-1}, 2, 2})\gamma \\
&\quad + (a_{j_1, \dots, j_{r-1}, 1, 2} - a_{j_1, \dots, j_{r-1}, 2, 1}) E_1 \cdot E_2 \cdot \gamma \\
&\quad + b_{j_1, \dots, j_{r-1}, 2} E_1 \cdot \gamma - b_{j_1, \dots, j_{r-1}, 1} E_2 \cdot \gamma
\end{aligned} \tag{20}$$

for all  $j_1, \dots, j_{r-1} \in \{1, 2\}$ . We conclude  $b_{j_1, \dots, j_r} = 0$  for all  $j_1, \dots, j_r \in \{1, 2\}$ . Next consider  $a_{j_1, \dots, j_r, i}$  with fixed  $j_1, \dots, j_r, i \in \{1, 2\}$ . If we have  $j_k = i$  for all  $k \in \{1, \dots, r\}$ , then by (19) we know that  $a_{j_1, \dots, j_r, i} = 0$ . If there exists  $k$  such that  $j_k \neq i$  it follows from the coefficient of  $E_1 \cdot E_2 \cdot \gamma$  in (20) that

$$a_{i, j_1, \dots, \widehat{j_k}, \dots, j_r, j_k} = a_{j_1, \dots, j_r, i}.$$

Again (19) yields  $a_{j_1, \dots, j_r, i} = 0$ . We conclude that all  $a_{j_1, \dots, j_r, i}$  vanish and that  $\nabla^r A\psi(0) = 0$ . This proves the assertion in the case  $n = 2$ .

Next consider  $n = 3$ . In the Bourguignon-Gauduchon trivialization we have

$$\begin{aligned}
A\psi(x) &= \frac{1}{r!} \sum_{j_1, \dots, j_r=1}^3 x_{j_1} \dots x_{j_r} \nabla_{E_{j_1}} \dots \nabla_{E_{j_r}} A\psi(0) \\
&\quad + \frac{1}{(r+1)!} \sum_{j_1, \dots, j_r, i=1}^3 x_{j_1} \dots x_{j_r} x_i \nabla_{E_{j_1}} \dots \nabla_{E_{j_r}} \nabla_{E_i} A\psi(0) \\
&\quad + o(|x|^{r+1})
\end{aligned}$$

by Taylor's formula and

$$A(G_\lambda^g(\cdot, p)\varphi)(x) = -\frac{1}{4\pi|x|^3} x \cdot \gamma + \frac{\lambda}{4\pi|x|} \gamma + o(|x|^{-s})$$



for every  $s > 0$  by Theorem 3.11, where  $\gamma$  is as above. It follows that

$$\begin{aligned}
0 &= -4\pi r!|x|^3 \text{Re}\langle A(G_\lambda^g(\cdot, p)\varphi)(x), A\psi(x) \rangle \\
&= \sum_{i,j_1,\dots,j_r=1}^3 x_{j_1}\dots x_{j_r} x_i \text{Re}\langle E_i \cdot \gamma, \nabla_{E_{j_1}} \dots \nabla_{E_{j_r}} A\psi(0) \rangle \\
&\quad + \frac{1}{r+1} \sum_{i,j_1,\dots,j_r,m=1}^3 x_{j_1}\dots x_{j_r} x_i x_m \text{Re}\langle E_i \cdot \gamma, \nabla_{E_{j_1}} \dots \nabla_{E_{j_r}} \nabla_{E_m} A\psi(0) \rangle \\
&\quad - \lambda \sum_{j_1,\dots,j_r=1}^3 x_{j_1}\dots x_{j_r} |x|^2 \text{Re}\langle \gamma, \nabla_{E_{j_1}} \dots \nabla_{E_{j_r}} A\psi(0) \rangle + o(|x|^{r+2}). \quad (21)
\end{aligned}$$

From the first term on the right hand side we obtain

$$\begin{aligned}
0 &= \text{Re}\langle E_i \cdot \gamma, \nabla_{E_{j_1}} \dots \nabla_{E_{j_r}} A\psi(0) \rangle \\
&\quad + \sum_{s=1}^r \text{Re}\langle E_{j_s} \cdot \gamma, \nabla_{E_i} \nabla_{E_{j_1}} \dots \widehat{\nabla_{E_{j_s}}} \dots \nabla_{E_{j_r}} A\psi(0) \rangle. \quad (22)
\end{aligned}$$

for all  $j_1, \dots, j_r, i \in \{1, 2, 3\}$ , where the hat means that the operator is left out. Our next aim is to obtain an analogue of (18) from the second and third term on the right hand side of (21). It is more difficult than in the case  $n = 2$ , since derivatives of both orders  $r$  and  $r + 1$  appear. The equation (8) reads

$$\begin{aligned}
\lambda A\psi &= D^{g_{\text{eucl}}} A\psi + \sum_{i,j=1}^3 (B_i^j - \delta_i^j) E_i \cdot \nabla_{E_j} A\psi \\
&\quad + \frac{1}{4} \sum_{i,j,k=1}^3 \tilde{\Gamma}_{ij}^k E_i \cdot E_j \cdot E_k \cdot A\psi.
\end{aligned}$$

Using (9), (10) and that  $|A\psi(x)|_{g_{\text{eucl}}} = O(|x|^r)$  as  $x \rightarrow 0$  we find

$$\lambda A\psi = D^{g_{\text{eucl}}} A\psi + O(|x|^{r+1})$$

and therefore

$$\nabla_{E_{j_1}} \dots \nabla_{E_{j_r}} D^{g_{\text{eucl}}} A\psi(0) = \lambda \nabla_{E_{j_1}} \dots \nabla_{E_{j_r}} A\psi(0) \quad (23)$$

for all  $j_1, \dots, j_r \in \{1, 2, 3\}$ . Using the equation (7) we find

$$A\nabla_{e_i}^g \psi = \nabla_{E_i} A\psi + O(|x|^{r+1}), \quad A\nabla_{e_i}^g \nabla_{e_j}^g \psi = \nabla_{E_i} \nabla_{E_j} A\psi + O(|x|^r)$$

for all  $i, j \in \{1, 2, 3\}$ . Since by definition  $d\rho|_x^{-1}(e_i) = E_i + O(|x|^2)$  the second term in the local formula (5) for  $\nabla^* \nabla$  vanishes at  $p$  and therefore we find

$$A \nabla^* \nabla \psi = - \sum_{i=1}^3 \nabla_{E_i} \nabla_{E_i} A \psi + O(|x|^r).$$

From the Schrödinger-Lichnerowicz formula (6) it follows that

$$\lambda^2 A \psi - \frac{\text{scal}}{4} A \psi = - \sum_{i=1}^3 \nabla_{E_i} \nabla_{E_i} A \psi + O(|x|^r)$$

and thus

$$\nabla_{E_{j_1}} \dots \nabla_{E_{j_{r-1}}} \sum_{i=1}^3 \nabla_{E_i} \nabla_{E_i} A \psi(0) = 0 \quad (24)$$

for all  $j_1, \dots, j_{r-1} \in \{1, 2, 3\}$ . Now recall the second and third term on the right hand side of (21)

$$\begin{aligned} 0 &= \frac{1}{r+1} \sum_{j_1, \dots, j_r, i, m=1}^3 x_{j_1} \dots x_{j_r} x_i x_m \text{Re} \langle E_i \cdot \gamma, \nabla_{E_{j_1}} \dots \nabla_{E_{j_r}} \nabla_{E_m} A \psi(0) \rangle \\ &\quad - \lambda \sum_{j_1, \dots, j_r=1}^3 x_{j_1} \dots x_{j_r} |x|^2 \text{Re} \langle \gamma, \nabla_{E_{j_1}} \dots \nabla_{E_{j_r}} A \psi(0) \rangle \end{aligned}$$

and let  $k_1, k_2, k_3 \in \mathbb{N}$  such that  $k_1 + k_2 + k_3 = r$ . Then from the coefficient of  $x_1^{k_1+2} x_2^{k_2} x_3^{k_3}$  we find

$$0 = \text{Re} \langle E_1 \cdot \gamma, \nabla_{E_1}^{k_1} \nabla_{E_2}^{k_2} \nabla_{E_3}^{k_3} \nabla_{E_1} A \psi(0) \rangle \frac{r!}{(k_1+1)!k_2!k_3!} \quad (I)$$

$$+ \text{Re} \langle E_2 \cdot \gamma, \nabla_{E_1}^{k_1} \nabla_{E_2}^{k_2-1} \nabla_{E_3}^{k_3} \nabla_{E_1}^2 A \psi(0) \rangle \frac{r!k_2}{(k_1+2)!k_2!k_3!} \quad (III)$$

$$+ \text{Re} \langle E_3 \cdot \gamma, \nabla_{E_1}^{k_1} \nabla_{E_2}^{k_2} \nabla_{E_3}^{k_3-1} \nabla_{E_1}^2 A \psi(0) \rangle \frac{r!k_3}{(k_1+2)!k_2!k_3!} \quad (IV)$$

$$- \lambda \text{Re} \langle \gamma, \nabla_{E_1}^{k_1} \nabla_{E_2}^{k_2} \nabla_{E_3}^{k_3} A \psi(0) \rangle \frac{r!}{k_1!k_2!k_3!} \quad (V)$$

$$- \lambda \text{Re} \langle \gamma, \nabla_{E_1}^{k_1} \nabla_{E_2}^{k_2-2} \nabla_{E_3}^{k_3} \nabla_{E_1}^2 A \psi(0) \rangle \frac{r!k_2(k_2-1)}{(k_1+2)!k_2!k_3!} \quad (VII)$$

$$- \lambda \text{Re} \langle \gamma, \nabla_{E_1}^{k_1} \nabla_{E_2}^{k_2} \nabla_{E_3}^{k_3-2} \nabla_{E_1}^2 A \psi(0) \rangle \frac{r!k_3(k_3-1)}{(k_1+2)!k_2!k_3!} \quad (VIII).$$

From the coefficient of  $x_1^{k_1} x_2^{k_2+2} x_3^{k_3}$  we find

$$0 = \text{Re}\langle E_1 \cdot \gamma, \nabla_{E_1}^{k_1-1} \nabla_{E_2}^{k_2} \nabla_{E_3}^{k_3} \nabla_{E_2}^2 A\psi(0) \rangle \frac{r!k_1}{k_1!(k_2+2)!k_3!} \quad (II)$$

$$+ \text{Re}\langle E_2 \cdot \gamma, \nabla_{E_1}^{k_1} \nabla_{E_2}^{k_2} \nabla_{E_3}^{k_3} \nabla_{E_2} A\psi(0) \rangle \frac{r!}{k_1!(k_2+1)!k_3!} \quad (I)$$

$$+ \text{Re}\langle E_3 \cdot \gamma, \nabla_{E_1}^{k_1} \nabla_{E_2}^{k_2} \nabla_{E_3}^{k_3-1} \nabla_{E_2}^2 A\psi(0) \rangle \frac{r!k_3}{k_1!(k_2+2)!k_3!} \quad (IV)$$

$$- \lambda \text{Re}\langle \gamma, \nabla_{E_1}^{k_1-2} \nabla_{E_2}^{k_2} \nabla_{E_3}^{k_3} \nabla_{E_2}^2 A\psi(0) \rangle \frac{r!k_1(k_1-1)}{k_1!(k_2+2)!k_3!} \quad (VI)$$

$$- \lambda \text{Re}\langle \gamma, \nabla_{E_1}^{k_1} \nabla_{E_2}^{k_2} \nabla_{E_3}^{k_3} A\psi(0) \rangle \frac{r!}{k_1!k_2!k_3!} \quad (V)$$

$$- \lambda \text{Re}\langle \gamma, \nabla_{E_1}^{k_1} \nabla_{E_2}^{k_2} \nabla_{E_3}^{k_3-2} \nabla_{E_2}^2 A\psi(0) \rangle \frac{r!k_3(k_3-1)}{k_1!(k_2+2)!k_3!} \quad (VIII).$$

From the coefficient of  $x_1^{k_1} x_2^{k_2} x_3^{k_3+2}$  we find

$$0 = \text{Re}\langle E_1 \cdot \gamma, \nabla_{E_1}^{k_1-1} \nabla_{E_2}^{k_2} \nabla_{E_3}^{k_3} \nabla_{E_3}^2 A\psi(0) \rangle \frac{r!k_1}{k_1!k_2!(k_3+2)!} \quad (II)$$

$$+ \text{Re}\langle E_2 \cdot \gamma, \nabla_{E_1}^{k_1} \nabla_{E_2}^{k_2-1} \nabla_{E_3}^{k_3} \nabla_{E_3}^2 A\psi(0) \rangle \frac{r!k_2}{k_1!k_2!(k_3+2)!} \quad (III)$$

$$+ \text{Re}\langle E_3 \cdot \gamma, \nabla_{E_1}^{k_1} \nabla_{E_2}^{k_2} \nabla_{E_3}^{k_3} \nabla_{E_3} A\psi(0) \rangle \frac{r!}{k_1!k_2!(k_3+1)!} \quad (I)$$

$$- \lambda \text{Re}\langle \gamma, \nabla_{E_1}^{k_1-2} \nabla_{E_2}^{k_2} \nabla_{E_3}^{k_3} \nabla_{E_3}^2 A\psi(0) \rangle \frac{r!k_1(k_1-1)}{k_1!k_2!(k_3+2)!} \quad (VI)$$

$$- \lambda \text{Re}\langle \gamma, \nabla_{E_1}^{k_1} \nabla_{E_2}^{k_2-2} \nabla_{E_3}^{k_3} \nabla_{E_3}^2 A\psi(0) \rangle \frac{r!k_2(k_2-1)}{k_1!k_2!(k_3+2)!} \quad (VII)$$

$$- \lambda \text{Re}\langle \gamma, \nabla_{E_1}^{k_1} \nabla_{E_2}^{k_2} \nabla_{E_3}^{k_3} A\psi(0) \rangle \frac{r!}{k_1!k_2!k_3!} \quad (V).$$

We multiply the first equation with  $\frac{(k_1+2)!k_2!k_3!}{r!}$ , the second equation with  $\frac{k_1!(k_2+2)!k_3!}{r!}$  and the third equation with  $\frac{k_1!k_2!(k_3+2)!}{r!}$  and then add the multiplied equations. If we consider the lines with the same Roman numbers

separately and use (23), (24), then we find

$$\begin{aligned}
0 &= -2\lambda \text{Re}\langle \gamma, \nabla_{E_1}^{k_1} \nabla_{E_2}^{k_2} \nabla_{E_3}^{k_3} A\psi(0) \rangle \\
&\quad + \text{Re}\langle E_1 \cdot \gamma, \nabla_{E_1}^{k_1} \nabla_{E_2}^{k_2} \nabla_{E_3}^{k_3} \nabla_{E_1} A\psi(0) \rangle k_1 \\
&\quad + \text{Re}\langle E_2 \cdot \gamma, \nabla_{E_1}^{k_1} \nabla_{E_2}^{k_2} \nabla_{E_3}^{k_3} \nabla_{E_2} A\psi(0) \rangle k_2 \\
&\quad + \text{Re}\langle E_3 \cdot \gamma, \nabla_{E_1}^{k_1} \nabla_{E_2}^{k_2} \nabla_{E_3}^{k_3} \nabla_{E_3} A\psi(0) \rangle k_3 \tag{I} \\
&\quad - \text{Re}\langle E_1 \cdot \gamma, \nabla_{E_1}^{k_1} \nabla_{E_2}^{k_2} \nabla_{E_3}^{k_3} \nabla_{E_1} A\psi(0) \rangle k_1 \tag{II} \\
&\quad - \text{Re}\langle E_2 \cdot \gamma, \nabla_{E_1}^{k_1} \nabla_{E_2}^{k_2} \nabla_{E_3}^{k_3} \nabla_{E_2} A\psi(0) \rangle k_2 \tag{III} \\
&\quad - \text{Re}\langle E_3 \cdot \gamma, \nabla_{E_1}^{k_1} \nabla_{E_2}^{k_2} \nabla_{E_3}^{k_3} \nabla_{E_3} A\psi(0) \rangle k_3 \tag{IV} \\
&\quad - \lambda \text{Re}\langle \gamma, \nabla_{E_1}^{k_1} \nabla_{E_2}^{k_2} \nabla_{E_3}^{k_3} A\psi(0) \rangle \sum_{i=1}^3 (k_i + 2)(k_i + 1) \tag{V} \\
&\quad + \lambda \text{Re}\langle \gamma, \nabla_{E_1}^{k_1} \nabla_{E_2}^{k_2} \nabla_{E_3}^{k_3} A\psi(0) \rangle k_1(k_1 - 1) \tag{VI} \\
&\quad + \lambda \text{Re}\langle \gamma, \nabla_{E_1}^{k_1} \nabla_{E_2}^{k_2} \nabla_{E_3}^{k_3} A\psi(0) \rangle k_2(k_2 - 1) \tag{VII} \\
&\quad + \lambda \text{Re}\langle \gamma, \nabla_{E_1}^{k_1} \nabla_{E_2}^{k_2} \nabla_{E_3}^{k_3} A\psi(0) \rangle k_3(k_3 - 1) \tag{VIII}.
\end{aligned}$$

Therefore we obtain the analogue of (18) namely

$$\text{Re}\langle \gamma, \nabla_{E_{j_1}} \dots \nabla_{E_{j_r}} A\psi(0) \rangle = 0$$

for all  $j_1, \dots, j_r \in \{1, 2, 3\}$ . Thus there exist  $a_{j_1, \dots, j_r, k} \in \mathbb{R}$  such that

$$\nabla_{E_{j_1}} \dots \nabla_{E_{j_r}} A\psi(0) = \sum_{k=1}^3 a_{j_1, \dots, j_r, k} E_k \cdot \gamma.$$

Observe that the coefficients  $a_{j_1, \dots, j_r, k}$  are symmetric in the first  $r$  indices. We insert this into (22) and we obtain

$$0 = a_{j_1, \dots, j_r, i} + \sum_{k=1}^r a_{i, j_1, \dots, \widehat{j_k}, \dots, j_r, j_k} \tag{25}$$

for all  $j_1, \dots, j_r, i \in \{1, 2, 3\}$ . On the other hand since  $\psi \in \ker(D^g - \lambda)$  we

find using the induction hypothesis

$$\begin{aligned}
0 &= \lambda \nabla_{E_{j_1}} \dots \nabla_{E_{j_{r-1}}} A\psi(0) \\
&= \nabla_{E_{j_1}} \dots \nabla_{E_{j_{r-1}}} \sum_{i=1}^3 E_i \cdot \nabla_{E_i} A\psi(0) \\
&= \sum_{i,k=1}^3 a_{j_1, \dots, j_{r-1}, i, k} E_i \cdot E_k \cdot \gamma \\
&= - \sum_{i=1}^3 a_{j_1, \dots, j_{r-1}, i, i} \gamma \\
&\quad + \sum_{\substack{i,k=1 \\ i < k}}^3 (a_{j_1, \dots, j_{r-1}, i, k} - a_{j_1, \dots, j_{r-1}, k, i}) E_i \cdot E_k \cdot \gamma \tag{26}
\end{aligned}$$

for all  $j_1, \dots, j_{r-1} \in \{1, 2, 3\}$ . Consider  $a_{j_1, \dots, j_r, i}$  with  $j_1, \dots, j_r, i \in \{1, 2, 3\}$ . If  $j_k = i$  for all  $k \in \{1, \dots, r\}$  then by (25) we know that  $a_{j_1, \dots, j_r, i} = 0$ . If there exists  $k$  such that  $j_k \neq i$  it follows from the coefficient of  $E_{j_k} \cdot E_i \cdot \gamma$  in (26) that

$$a_{i, j_1, \dots, \widehat{j_k}, \dots, j_r, j_k} = a_{j_1, \dots, j_r, i}.$$

Again (25) yields  $a_{j_1, \dots, j_r, i} = 0$ . We conclude that all  $a_{j_1, \dots, j_r, i}$  vanish and that  $\nabla^r A\psi(0) = 0$ . This proves the assertion in the case  $n = 3$ .  $\square$

**Remark 4.9.** *It is not clear how to prove this lemma for  $n \geq 4$ . Namely the condition  $\text{Re}\langle \gamma, \nabla_{E_i} A\psi(0) \rangle = 0$  for all  $i$  leads to*

$$\nabla_{E_i} A\psi(0) = \sum_{k=1}^n a_{ik} E_k \cdot \gamma + \sum_{k=1}^n b_{ik} \cdot \gamma$$

with  $a_{ik} \in \mathbb{R}$  and  $b_{ik} \in \text{Cl}(n)$ . As above it follows that  $a_{ik} = -a_{ki}$  for all  $i, k$  and furthermore

$$\begin{aligned}
0 &= \lambda A\psi(0) \\
&= \sum_{i=1}^n E_i \cdot \nabla_{E_i} A\psi(0) \\
&= 2 \sum_{\substack{i,k=1 \\ i < k}}^n a_{ik} E_i \cdot E_k \cdot \gamma + \sum_{i,k=1}^n E_i \cdot b_{ik} \cdot \gamma.
\end{aligned}$$

But for  $n \geq 4$  the spinors  $E_1 \cdot E_2 \cdot \gamma$  and  $E_3 \cdot E_4 \cdot \gamma$  are not linearly independent in general. Thus we cannot conclude immediately that all the  $a_{ik}$  vanish.

*Proof of Theorem 4.4.* Let  $k, m \in \mathbb{N} \setminus \{0\}$ , let  $g \in R(M)$  and equip  $[g]$  with the  $C^k$ -topology. Let  $U \subset [g]$  be open. In order to show that  $N_m(M) \cap [g]$  is dense in  $[g]$  we have to show that  $U \cap N_m(M)$  is not empty. Since  $S_m(M) \cap [g]$  is dense in  $[g]$ , there exists a metric in  $U \cap S_m(M)$ , which we denote again by  $g$ . Let  $\lambda$  be one of the eigenvalues  $\{\lambda_1^\pm, \dots, \lambda_m^\pm\}$  of  $D^g$  and let  $\psi$  be an eigenspinor corresponding to  $\lambda$ . Choose an open neighborhood  $V \subset [g]$  of  $g$  which is contained in  $U \cap S_m(M)$  and define

$$F_\psi : V \rightarrow C^\infty(\Sigma^g M)$$

as in Lemma 4.5

Next we show that a suitable restriction of  $F_\psi^{ev}$  is transverse to the zero section of  $\Sigma^g M$ . Let  $p \in M$  with  $\psi(p) = 0$ . By Lemma 4.8 there exists an open neighborhood  $U_p \subset M$  of  $p$  and  $f_{p,1}, \dots, f_{p,4} \in C^\infty(M, \mathbb{R})$  and an open neighborhood  $V_p \subset V_{f_{p,1} \dots f_{p,4}}$  of  $g$  such that for all  $(h, q) \in V_p \times U_p$  we have

$$\pi_1(dF_\psi^{ev}|_{(h,q)}(T_h V_{f_{p,1} \dots f_{p,4}} \oplus \{0\})) = \Sigma_q^g M. \quad (27)$$

Since the zero set of  $\psi$  is compact, there exist points  $p_1, \dots, p_r \in M$  and open neighborhoods  $U_{p_i} \subset M$  of  $p_i$  and  $f_{p_i,1}, \dots, f_{p_i,4} \in C^\infty(M, \mathbb{R})$  and open neighborhoods  $V_{p_i} \subset V_{f_{p_i,1} \dots f_{p_i,4}}$  of  $g$ ,  $1 \leq i \leq r$ , such that the open neighborhoods  $U_{p_1}, \dots, U_{p_r}$  cover the zero set of  $\psi$  and such that for every  $i$  the equation (27) holds for all  $(h, q) \in V_{p_i} \times U_{p_i}$ . We label the functions  $f_{p_i,j}$  by  $f_1, \dots, f_{4r}$  and define

$$V_{f_1 \dots f_{4r}} := \left\{ \left( 1 + \sum_{i=1}^{4r} t_i f_i \right) g \mid t_i \in \mathbb{R} \right\} \cap V.$$

and  $F_\psi : V_{f_1 \dots f_{4r}} \rightarrow C^\infty(\Sigma^g M)$  as in Lemma 4.5. Since  $M$  is compact there exists  $C > 0$  such that  $|\psi|_g \geq C$  on the complement of the union of the sets  $U_{p_i}$ . Thus we can find an open neighborhood  $V_\psi \subset V_{f_1 \dots f_{4r}}$  of  $g$  such that the equation (27) holds for all  $(h, q) \in V_\psi \times M$ . It follows that the restriction of  $F_\psi^{ev}$  to  $V_\psi \times M$  is transverse to the zero section of  $\Sigma^g M$ .

Define  $W_\psi$  as the subset of all  $h \in V_\psi$  such that  $F_\psi(h)$  is nowhere zero on  $M$ . By Remark 4.6 this condition is equivalent to the condition that  $F_\psi(h)$  is transverse to the zero section of  $\Sigma^g M$ . By Theorem 2.3 the set  $W_\psi$  is dense in  $V_\psi$ . Since the zero section is closed in  $\Sigma^g M$  and  $F_\psi^{ev}$  is continuous, the set  $W_\psi$  is also open in  $V_\psi$ . If  $h \in W_\psi$ , then the eigenspinor  $F_\psi(h)$  of  $D^{g,h}$  is nowhere zero on  $M$  and it corresponds to a simple eigenvalue of  $D^{g,h}$ . Thus all the eigenspinors of  $D^{g,h}$  corresponding to this eigenvalue are nowhere zero on  $M$ .

For every one of the finitely many eigenvalues  $\lambda_1^\pm, \dots, \lambda_m^\pm$  of  $D^g$  we choose an eigenspinor  $\psi$  and obtain an open subset  $W_\psi \subset V$  as above. Let  $W$  be

the intersection of these open subsets  $W_\psi$ . It is not empty, since the  $W_\psi$  are dense in a neighborhood of  $g$ . If  $h \in W$ , then all the eigenspinors of  $D^h$  corresponding to the eigenvalues  $\lambda_1^\pm, \dots, \lambda_m^\pm$  of  $D^h$  are nowhere zero on  $M$ . Since  $W \subset U$  by construction, we have  $U \cap N_m(M) \neq \emptyset$ . Thus  $N_m(M) \cap [g]$  is dense in  $[g]$ . We have already seen that  $N_m(M) \cap [g]$  is open in  $[g]$ .  $\square$

## 4.2 Examples on closed surfaces

In this section we give a counterexample showing that Theorem 1.1 does not hold for harmonic spinors in the case  $n = 2$ . Let  $(M, g, \Theta)$  be a closed Riemannian spin manifold of dimension 2. The spinor bundle splits as

$$\Sigma^g M = \Sigma^+ M \oplus \Sigma^- M$$

and sections of  $\Sigma^\pm M$  will be called positive respectively negative spinors. The manifold  $(M, g)$  is Kähler and the bundle  $\Sigma^+ M$  is canonically isomorphic to a holomorphic line bundle  $L$  on  $M$ . Furthermore positive harmonic spinors can be identified with holomorphic sections of  $L$  (see e. g. [Hit], [Bä2]).

To every positive or negative spinor on  $(M, g)$  one can associate a tangent vector field on  $M$  by a method given in [Am1] which we briefly recall. First we define  $\tau_\pm: \text{SO}(2) \rightarrow \mathbb{C}$  by

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \mapsto \exp(\pm it).$$

We define a complex structure  $J$  on  $M$  such that for every  $p \in M$  and for every unit vector  $X \in T_p M$  the system  $(X, JX)$  is a positively oriented orthonormal basis of  $T_p M$ . Then the map  $\text{P}_{\text{SO}}(M, g) \times_{\tau_+} \mathbb{C} \rightarrow (TM, J)$  which sends  $[(e_1, e_2), 1]$  to  $e_1$  is an isomorphism of complex line bundles. In the same way we obtain an isomorphism  $\text{P}_{\text{SO}}(M, g) \times_{\tau_-} \mathbb{C} \rightarrow (TM, -J)$ . Then the following holds.

**Lemma 4.10** ([Am1]). *Let  $(M, g, \Theta)$  be a Riemannian spin manifold of dimension 2. Then the map*

$$\begin{aligned} \Phi_\pm : \quad \Sigma^\pm M = \text{P}_{\text{Spin}}(M, g) \times_\rho \Sigma_2^\pm &\rightarrow \text{P}_{\text{SO}}(M, g) \times_{\tau_\mp} \mathbb{C} \cong (TM, \mp J) \\ [s, \sigma] &\mapsto [\Theta(s), \sigma^2] \end{aligned}$$

*is well defined.*

We denote by  $\gamma$  the genus of  $M$ . Assume that  $\psi$  is a nontrivial positive harmonic spinor on  $M$  and that  $p \in M$  is a point with  $\psi(p) = 0$ . After a choice of a local holomorphic chart of  $M$  and of a local trivialization of the holomorphic line bundle  $\Sigma^+M$  around  $p$  the spinor  $\psi$  corresponds locally to a holomorphic function. We define  $m_p$  as the order of the zero  $p$ . Let  $X$  be the vector field on  $M$  associated to  $\psi$  via Lemma 4.10. It follows that  $X$  has an isolated zero at  $p$  with index equal to  $-2m_p$ . Let  $\chi(M) = 2 - 2\gamma$  denote the Euler characteristic of  $M$ . Denote by  $N$  the zero set of  $\psi$ . Since  $M$  is compact, the set  $N$  is finite. By the Poincaré-Hopf Theorem we obtain the following result.

**Theorem 4.11.** *Assume that  $\psi$  is a positive harmonic spinor on a closed surface  $(M, g, \Theta)$  and let  $N \subset M$  be its zero set. Then  $N$  is finite and we have*

$$\sum_{p \in N} m_p = -\frac{1}{2}\chi(M) = \gamma - 1.$$

It follows from [Hit], Proposition 2.3, that on a closed oriented surface  $M$  of genus 2 there are exactly 6 distinct spin structures, such that for every Riemannian metric  $g$  on  $M$  we have  $\dim_{\mathbb{C}} \ker(D^g) = 2$ . We take one of these spin structures. By Theorem 4.11 for every choice of metric every positive harmonic spinor vanishes at exactly one point. Thus Theorem 1.1 does not hold for harmonic spinors in the case  $n = 2$ .

## References

- [Am1] B. Ammann, *Spin-Strukturen und das Spektrum des Dirac-Operators*. Dissertation zur Erlangung des Doktorgrades, Freiburg im Breisgau (1998).
- [Am2] B. Ammann, *A variational problem in conformal spin geometry*. Habilitationsschrift, Universität Hamburg (2003).
- [Am4] B. Ammann, *The smallest Dirac eigenvalue in a spin-conformal class and cmc-immersions*. Comm. Anal. Geom. **17**, 429-479 (2009).
- [AH] B. Ammann, E. Humbert, *Positive mass theorem for the Yamabe problem on spin manifolds*. GAFA. **15**, 567-576 (2005).
- [AGHM] B. Ammann, J.-F. Grosjean, E. Humbert, B. Morel, *A spinorial analogue of Aubin's inequality*. Math. Z. **260**, 127-151 (2008).



- [Ar] N. Aronszajn, *A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order*. J. Math. Pures Appl. **36**, 235-249 (1957).
- [AS] M. Abramowitz, I. Stegun (eds.), *Handbook of mathematical functions*. Dover Publications, New York (1964).
- [Bä1] C. Bär, *Lower eigenvalue estimates for Dirac operators*. Math. Ann. **293**, 39-46 (1992).
- [Bä2] C. Bär, P. Schmutz, *Harmonic spinors on Riemann surfaces*. Ann. Glob. Anal. Geom. **10**, 263-273 (1992).
- [Bä3] C. Bär, *The Dirac operator on space forms of positive curvature*. J. Math. Soc. Japan **48**, 69-83 (1996).
- [Bä3] C. Bär, *On nodal sets for Dirac and Laplace operators*. Comm. Math. Phys. **188**, 3, 709-721 (1997).
- [BG] J.-P. Bourguignon, P. Gauduchon, *Spineurs, opérateurs de Dirac et variations de métriques*. Comm. Math. Phys. **144**, 3, 581-599 (1992).
- [Da] M. Dahl, *Dirac eigenvalues for generic metrics on three-manifolds*. Ann. Glob. Anal. Geom. **24**, 1, 95100 (2003).
- [DM] A. Dimakis, F. Müller-Hoissen, *On a gauge condition for orthonormal three-frames*. Phys. Lett. A **142**, 73-74 (1989).
- [FNS] J. Frauendiener, J. M. Nester, L. B. Szabados, *Witten spinors on maximal, conformally flat hypersurfaces*. Class. Quantum Grav. **28**, 185004 (2011).
- [Hij1] O. Hijazi, *A conformal lower bound for the smallest eigenvalue of the Dirac operator and Killing spinors*. Comm. Math. Phys. **104**, 1, 151-162 (1986).
- [Hij2] O. Hijazi, *Première valeur propre de l'opérateur de Dirac et nombre de Yamabe*. C. R. Acad. Sci. Paris **313**, 12, 865-868 (1991).
- [Hir] M. Hirsch, *Differential topology*. Springer Verlag, New York (1976).
- [Hit] N. Hitchin, *Harmonic spinors*. Advances in Math. **14**, 1-55 (1974).
- [K] T. Kato, *Perturbation theory of linear operators*. Reprint of the 1980 edition, Springer Verlag, Berlin, Heidelberg (1995).

- [LM] H. B. Lawson, M.-L. Michelsohn, *Spin geometry*. Princeton University Press, Princeton (1989).
- [Lo] J. Lott, *Eigenvalue bounds for the Dirac operator*. Pac. J. Math. **125**, 1, 117-126 (1986).
- [LP] J. M. Lee, T. H. Parker, *The Yamabe problem*. Bull. Amer. Math. Soc., New Ser. **17**, 1, 37-91 (1987).
- [Ma] S. Maier, *Generic metrics and connections on Spin- and Spin<sup>c</sup>-manifolds*. Comm. Math. Phys. **188**, 2, 407-437 (1997).
- [N] J. M. Nester, *A positive gravitational energy proof*. Phys. Lett. A **139**, 112-114 (1989).
- [U] K. Uhlenbeck, *Generic properties of eigenfunctions*. Amer. J. Math. **98**, 4, 1059-1078 (1976).
- [Wi] E. Witten, *A new proof of the positive energy theorem*. Comm. Math. Phys. **80**, 3, 381-402 (1981).

**Author's address**

Andreas Hermann, Fakultät für Mathematik, Universität Regensburg,  
 93040 Regensburg, Germany  
 Email address: andreas.hermann@mathematik.uni-regensburg.de,  
 andreas\_hermann@gmx.de  
 URL: <http://homepages.uni-regensburg.de/~hea06979/>